Quantum back-action of variable-strength measurement

M. Hatridge\textsuperscript{1*} and S. Shankar,\textsuperscript{1} M. Mirrahimi,\textsuperscript{1,2} F. Schackert,\textsuperscript{1} K. Geerlings,\textsuperscript{1} T. Brecht,\textsuperscript{1} K. M. Sliwa,\textsuperscript{1} B. Abdo,\textsuperscript{1} L. Frunzio,\textsuperscript{1} S. M. Girvin,\textsuperscript{1} R. J. Schoelkopf,\textsuperscript{1} M. H. Devoret\textsuperscript{1}

\textsuperscript{1}Department of Applied Physics and Physics, Yale University, New Haven, Connecticut 06520, USA, 
\textsuperscript{2}INRIA Paris-Rocquencourt, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France 

\textsuperscript{*}To whom correspondence should be addressed; E-mail: michael.hatridge@yale.edu.

Measuring a quantum system can randomly perturb its state. The strength and nature of this back-action depends on the quantity which is measured. In a partial measurement performed by an ideal apparatus, quantum physics predicts that the system remains in a pure state whose evolution can be tracked perfectly from the measurement record. We demonstrate this property using a superconducting qubit dispersively coupled to a cavity traversed by a microwave signal. The back-action on the qubit state of a single measurement of both signal quadratures is observed and shown to produce a stochastic operation whose action is determined by the measurement result. This accurate
monitoring of a qubit state is an essential prerequisite for measurement-based feedback control of quantum systems.

While the behavior (‘state collapse’) of a quantum system subject to an infinitely-strong, i.e. projective, quantum non-demolition (QND) measurement is textbook physics, the subtlety and utility of finite-strength, i.e. partial, measurement phenomena are neither widely appreciated nor commonly verified experimentally. Standard quantum measurement theory puts forward the principle that observing a system induces a decoherent evolution proportional to the measurement strength (1–5). Thus, partial measurement is often associated with partial decoherence of the state of a quantum system. However, this measurement-induced degradation occurs only if the measurement is inefficient “informationally”, i.e. if only a portion of the measurement’s information content is available to the observer for use in reconstructing the new state of the system.

If, instead, the measurement apparatus is entirely efficient, the new state of the quantum system can be perfectly reconstructed. This outcome-dependent revision of the system’s imposed initial conditions constitutes a fundamental quantum effect called “measurement back-action” (2, 6–8). Although the system’s evolution under measurement is erratic, hence the measurement outcome cannot be predicted in advance, the measurement record faithfully reports the perturbation of the system after the fact.

We utilize the powerful, combined qubit-cavity architecture, Circuit Quantum Electrodynamics (cQED) (9, 10), which allows for rapid, repeated Quantum Non-Demolition (QND) (11, 12) superconducting qubit measurement (13–18). The cavity output is monitored in real time using a phase-preserving amplifier working near the quantum-limit, where the noise is only caused by the fundamental quantum fluctuations of the electrodynamic vacuum (19). The decision to read out our qubit using coherent states of the resonator has two important consequences. First, the outcomes of a partial measurement form a quasi-continuum, unlike the set of discrete
answers obtained from a projective measurement. Second, measuring both quadratures of the signal leads to two-dimensional diffusion of the direction of the qubit effective spin. We show that the choice of measurement apparatus and of measurement strength both affect the evolution of a quantum system, but neither results in degradation of the system’s state if the measurement is informationally efficient. Such precise knowledge of the measurement back-action is a necessary prerequisite for general feedback control of quantum systems.

Our superconducting qubit is a transmon (20), consisting of two Josephson junctions in a closed loop, shunted by a capacitor to form an anharmonic oscillator. The two lowest energy states, (\(|g\rangle\) and \(|e\rangle\)), are the logical states of the qubit. The qubit is dispersively coupled to a compact resonator, which is further asymmetrically coupled to input and output transmission lines (Fig. 1B, C), determining the resonator bandwidth (\(\kappa/2\pi = 5.8\) MHz). To measure a qubit prepared in initial state \(|\psi\rangle = c_g |g\rangle + c_e |e\rangle\), a microwave pulse of duration \(T_m \gg 1/\kappa\) is applied to the resonator. The state dependent shift of the resonator frequency (\(\chi/2\pi = 5.4\) MHz) results in an entangled state of the qubit and pulse \(|\Psi\rangle = c_g |g\rangle \otimes |\alpha_g\rangle + c_e |e\rangle \otimes |\alpha_e\rangle\), where \(|\alpha_g,e\rangle\) refer to the coherent state after traversing the resonator.

Amplification is required to convert the pointer state \(|\Psi\rangle\) into a macroscopic signal that can be processed and recorded with standard instrumentation. In our case, the pulse having traversed the resonator is amplified using a linear, phase-preserving amplifier with gain \(G\), which can be seen as multiplying the average photon number in \(|\alpha_g,e\rangle\)(see Fig. 1B). For dynamical range considerations, our Josephson amplifier is operated in this experiment with a gain \(G = 12.5\) dB and bandwidth of 6 MHz, adding close to the minimum amount of noise allowed by quantum mechanics (21–24). The added quantum fluctuations are due to a second, “idler”, input (19). A measurement of both quadratures of the output mode results in an outcome, denoted \((I_m, Q_m)\), which is then used to determine the new state of the qubit after measurement (see Fig. 1A). As has been shown in (8), and detailed in the supplementary material, this outcome contains
all information necessary to perfectly reconstruct the new state of the qubit. Remarkably, the additional quantum fluctuations introduced during amplification enter in the measurement back-action on the qubit without impairing our knowledge of it.

We first demonstrate projective qubit readout by strongly measuring the qubit using an 8 µs pulse with the drive power set so that the average number of photons in the resonator during the pulse was $\bar{n} = 5$ (Fig 2). Selected individual measurement records for the qubit are shown in Fig. 2B. The data are digitized with a sampling time of 20 ns and smoothed with a binomial filter with $T_m = 240$ ns width, corresponding to 8 cavity lifetimes, and scaled by the experimentally determined standard deviation ($\sigma$). The highlighted trace shows clear quantum jumps in the qubit state, which are identified by vertical black dotted lines indicating $4\sigma$ deviations from the current qubit state. The 8% equilibrium qubit excited state population is consistent with other measurements of superconducting qubits (25). By counting the number of up and down transitions in 25,000 traces with no qubit excitation pulse, we calculate $T_1 \leq 3.1$ µs. Although we fail to resolve pairs of transitions separated by much less than our filter time constant, this method for estimating $T_1$ yields a value in good agreement with the value $T_1 = 2.8$ µs calculated from fitting an exponential to the averaged trajectory of all traces. Further, the average qubit polarization did not vary over 8 µs of continuous measurement, nor did $T_1$ diminish with larger readout amplitude up to $\bar{n} \simeq 15$, demonstrating the QND nature of our readout.

Histograms of the scaled $I_m$ component of the outcome for the first 240 ns of measurement after a qubit rotation by $\theta = 0, \pi/2, \pi$ are shown in Fig. 2C. The ground and excited distributions are separated by 4.8 standard deviations, corresponding to a measurement fidelity of 98% when $I_m = 0$ is used as the discrimination threshold. We emphasize that the discreteness of the $z$ measurement of the transmon circuit, illustrated by the bimodality of the histogram, is here due only to the quantum nature of the circuit and not to any nonlinearity of the readout. Thus, this measurement of a continuous, unbounded pointer state is exactly equivalent to the
Stern-Gerlach experiment. These strong, high-fidelity measurements allow us to perform precise tomography, and to prepare the qubit in a known state by measurement. We next use these tools to quantify measurement back-action of partial measurement on the qubit state.

The qubit evolution due to partial measurement can be precisely calculated from the complete measurement record using the quantum trajectory approach (6, 7), but this is computationally intensive. Instead, we calculate the back-action from the average output over the time $T_m$, as in (8). Provided that the measurement time is short compared to the qubit coherence times $T_1$ and $T_2$, and long compared to the cavity lifetime and amplifier response time, this approach allows the qubit to be tracked without degradation. In this experiment, $T_m = 240$ ns, which is shorter than $T_1 = 2.8 \mu s$ and $T_{2R} = 0.7 - 2.0 \mu s$, and much longer than the cavity lifetime and JPC response time of 30 ns.

Assuming the qubit is initially polarized along +y-axis, we calculate the final qubit Bloch vector $(x_f, y_f, z_f)$ as a function of measurement outcome $(I_m, Q_m)$ (see detailed derivation in the Supplementary Materials) to be:

$$
x_f^q(I_m, Q_m) = \text{sech} \left( \frac{I_m \bar{I}_m}{\sigma^2} \right) \sin \left( \frac{Q_m \bar{I}_m}{\sigma^2} + \frac{\bar{Q}_m \bar{I}_m}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right) \right) e^{-\frac{I_m^2}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right)},
$$

$$
y_f^q(I_m, Q_m) = \text{sech} \left( \frac{I_m \bar{I}_m}{\sigma^2} \right) \cos \left( \frac{Q_m \bar{I}_m}{\sigma^2} + \frac{\bar{Q}_m \bar{I}_m}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right) \right) e^{-\frac{I_m^2}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right)},
$$

$$
z_f^q(I_m) = \tanh \left( \frac{I_m \bar{I}_m}{\sigma^2} \right),
$$

where $\bar{I}_m$ and $\bar{Q}_m$ and $\sigma$ define the center and standard deviation of the outcome distributions, and $\eta$ is the quantum efficiency of the amplification chain (Fig. 1A). In this theory, we neglect the effect of qubit decoherence and losses before amplification. In the limit of a perfectly efficient amplification ($\eta = 1$), we see that the length of the Bloch vector is unity, irrespective of outcome. The parameter $I_m/\sigma$ can be identified as the apparent measurement
strength since the measurement becomes more strongly projective as $\bar{I}_m/\sigma$ increases. It is given in terms of experimental parameters as $\bar{I}_m/\sigma = \sqrt{2\eta\kappa T_m} \sin(\theta/2)$, where $\theta = 2 \arctan \chi/\kappa$.

The pulse sequence for determining measurement back action is shown in Fig. 3A. We first strongly read out the qubit with a 240 ns, $\bar{n} = 5$ pulse, and record the outcome, which will be used to prepare the qubit in the ground state by post-selection. Then the qubit is rotated to the $+y$ axis and measured with a variable measurement strength ($T_m = 240$ ns), and the outcome ($I_m, Q_m$) recorded. The final, tomography, phase measures the $x$, $y$, or $z$ component of the qubit Bloch vector with a strong ($\bar{n} = 5$, $T_m = 240$ ns) measurement pulse. To compensate for the finite readout strength and qubit temperature, trials with outcomes $|I_m/\sigma| < 1.5$ (corresponding to state purity $< 99\%$) for the first and third measurements are discarded, as well as outcomes for the first measurement with the qubit in $|e\rangle$. To quantify the measurement back-action for a given measurement outcome ($I_m, Q_m$), the average final qubit Bloch vector, conditioned by the measurement outcome ($I_m, Q_m$), ($\langle X \rangle_c, \langle Y \rangle_c, \langle Z \rangle_c$), is calculated versus outcome using the results of the tomography phase. These conditional maps of $\langle X \rangle_c, \langle Y \rangle_c, \langle Z \rangle_c$ were constructed using 201 by 201 bins in the plane of scaled measurement outcomes ($I_m/\sigma, Q_m/\sigma$).

Results for four measurement strengths increasing by decades from $\bar{n} = 5 \times 10^{-3}$ to 5 are shown in Fig. 3B (see Supplementary Movie S1 of histograms and tomograms for all measurement strengths). The left column shows a two-dimensional histogram of all scaled measurement outcomes recorded during the variable strength readout pulse. At weak measurement strength, the ground (left) and excited (right) state distributions overlap almost completely. Their separation grows with increasing strength until they are well separated at $\bar{n} = 5$, corresponding to the strong projective measurement shown in Fig. 1A. The rightmost columns show $\langle X \rangle_c, \langle Y \rangle_c, \langle Z \rangle_c$ versus associated ($I_m/\sigma, Q_m/\sigma$) bin. At weak measurement strength ($\bar{n} \ll 1$), the qubit state is only slightly perturbed, with all measurement outcomes corresponding to Bloch vectors pointing nearly along the $+y$ (initial) axis. However, gradients in $\langle X \rangle_c$ along the $Q_m$-axis
and $\langle Z \rangle_c$ along the $I_m$-axis are visible, demonstrating the outcome-dependent back-action of the measurement on the qubit state. As the measurement strength increases, so does the back-action, as seen in the increase of the gradients in the $\langle X \rangle_c$ and $\langle Z \rangle_c$ maps (see Fig. S2). When the measurement becomes strong, the qubit is projected to $+z$ for positive $I_m$ ($-z$ for negative $I_m$) while $\langle X \rangle$ and $\langle Y \rangle$ go unconditionally to zero, as expected.

One of the key predictions of finite-strength measurement theory is that the statistics of the measurement process, in particular the apparent measurement strength in the I-quadrature (which can be determined experimentally from the statistics of the measurement outcomes), are sufficient to infer $\zeta$ for any apparent measurement strength or outcome (see Eq. (1)). For weak measurement, where the back-action is symmetric along both $x$ and $z$, the apparent measurement strength determines the amplitude of the $x$ back-action as well (see Supp. Mat. Eq. (14)). In Fig. 4A, we quantitatively compare this prediction with our experimental result. The scaling coefficients relating measurement outcome to back-action along $z$, $(\partial \langle Z \rangle_c / \partial I_m) \sigma$, and along $x$, $(\partial \langle X \rangle_c / \partial Q_m) \sigma$, extracted from the tomograms at $I_m = Q_m = 0$, are plotted versus the apparent measurement strength extracted from the histograms, $I_m / \sigma$ (see Supp. Mat. sec. 1.4).

Both coefficients, $((\partial \langle Z \rangle_c / \partial I_m) \sigma$ and $\langle \partial \langle X \rangle_c / \partial Q_m \rangle \sigma)$, are predicted at $I_m = Q_m = 0$ to be equal to $I_m / \sigma$; therefore the data in Fig. 4A should have unity slope. However, finite $T_1$ and $T_2$ acting for a time $\tau$ reduce the state purity and the apparent back-action. To first order, the coefficients are modified to $(\partial \langle Z \rangle_c / \partial I_m) \sigma = (\bar{I}_m / \sigma) e^{-\tau / T_1}$ and $(\partial \langle X \rangle_c / \partial Q_m) \sigma \simeq (\bar{I}_m / \sigma) e^{-\tau / T_2}$ for the $z$ and $x$ back-action, respectively. In our pulse sequence, $\tau \simeq 380$ ns, predicting slopes of $0.87 \pm 0.09$ and $0.58 \pm 0.06$ for $z$ and $x$, in excellent agreement with the experimentally determined slopes of $0.86 \pm 0.01$ and $0.55 \pm 0.01$. All further theoretical predictions are modified to reflect the effects of $T_1$ and $T_2$, following the description in Suppl. Mater. Eq. (2). The black curve is the full theoretical dependence of $(\partial \langle X \rangle_c / \partial Q_m) \sigma = \bar{I}_m / \sigma \cos \left( \bar{Q}_m \bar{I}_m / \sigma^2 \left( (1 - \eta) / \eta \right) \right) e^{-\bar{I}_m / \sigma^2 ((1-\eta)/\eta) e^{-\tau / T_2}}$ using $\eta = 0.2$, the lowest value of
we extract from other measurements (see Supp. Mat. sec. 1.3). We attribute the discrepancy between theory and data at high measurement strength to environmental dephasing effects due to finite $T_2$, and losses before the JPC. Additionally, we process the tomography results unconditioned by measurement outcome in Fig. 4B. Theory predicts $\langle Y \rangle = e^{-\bar{I}_m/\eta \sigma^2} \cos \left( \bar{I}_m \bar{Q}_m / \eta \sigma^2 \right) e^{-\tau/T_2}$. This expression evaluated with $\eta = 0.2$ is shown as a black curve with the deviation for stronger measurements attributed to dephasing effects due to losses before amplification.

Similar experiments have studied measurement of the state of a microwave cavity by Rydberg atoms (26), and partial nonlinear measurement of phase qubits (27). Also, phase-sensitive parametric amplification has been used to implement weak measurement-based feedback (18). In our experiment, the ability to perform both weak and strong high-efficiency, QND, linear measurements within a qubit lifetime, coupled with our high throughput and minimally noisy readout electronics, allow us to acquire 13.5 billion qubit measurements in approximately 28 hours, data which can be compared to complete theoretical predictions of the conditional evolution of quantum states under measurement. They provide strong evidence that the purity of the state would not decrease in the limit of a perfect measurement, even when the signal is processed by a phase-preserving amplifier.

Our experiment illustrates an alternate approach to the description of a quantum measurement. In the case of a qubit, a finite-strength QND measurement can be thought of as a stochastic operation whose action is unpredictable but known to the experimenters after the fact if they possess a quantum-noise-limited amplification chain. Any final state is possible, and the type of quantity measured, combined with the measurement strength, determines the probability distribution for different outcomes. This partial (i.e. finite-strength) measurement paradigm is not inconsistent with the usual view of projective (i.e. infinite-strength) measurement. Rather, projective measurement is the limiting case of the broader class of finite strength measurements.
The finite-strength measurement predictions that we have verified have immediate applicability to proposed schemes for feedback stabilization and error correction of superconducting qubit states. While classical feedback is predicated on the idea that measuring a system does not disturb it, quantum feedback has to make additional corrections to the state of the system to counteract the unavoidable measurement back-action. The measurement back-action that is the subject of this paper thus crucially determines the transformation of the measurement outcome into the optimal correction signal for feedback. Our ability to experimentally quantify the back-action of an arbitrary-strength measurement thus provides a dress rehearsal for full feedback control of a general quantum system.

References and Notes


28. The authors wish to thank Alexander Korotkov, Benjamin Huard, Matthew Reed, Andreas Wallraff and Christopher Eichler for helpful discussions. Facilities use was supported by the Yale Institute for Nanoscience and Quantum Engineering (YINQE) and the NSF MRSEC DMR 1119826. This research was supported in part by the Office of the Director of National Intelligence (ODNI), Intelligence Advanced Research Projects Activity (IARPA), through the Army Research Office (W911NF-09-1-0369) and in part by the U.S. Army Research Office (W911NF-09-1-0514). All statements of fact, opinion or conclusions contained herein are those of the authors and should not be construed as representing the official views or policies of IARPA, the ODNI, or the U.S. Government. M. M. acknowledges partial support from the Agence National de Recherche under the project EPOQ2, ANR-09-JCJC-0070. M. H. D. acknowledges partial support from the College de France. S. M. G. acknowledges support from the NSF DMR 1004406.
Josephson junctions

transmon

\( f_q = 5.676 \text{ GHz} \)

\( E_C = 290 \text{ MHz} \)

\( T_1 = 2.8 \text{ ms} \)

\( T_{2R} = 2.0 - 0.7 \text{ ms} \)

\( \frac{c}{2\pi} = 5.4 \text{ MHz} \)

\( Q_{\text{IN}} \sim 100,000 \)

\( Q_{\text{OUT}} = 1400 \)

measurement of

\((I_m, Q_m)\)

with probability

\( P(I_m, Q_m) \)

\( S_f = (x_f, y_f, z_f) \)

\( S_i = (x_i, y_i, z_i) \)

1 to 1 map

from \((I_m, Q_m)\) to \(S_f\)

quantum noise

readout pulse

readout pulse

readout pulse

Ref

Ref
Fig. 1. (A) Bloch sphere representation of the effect on the qubit state of a phase-preserving measurement in a cQED architecture. After a measurement with outcome \((I_m, Q_m)\), the qubit will be found in a final state \(\vec{S}_f = (x_f, y_f, z_f)\), with \(I_m\) encoding information on the projection of the qubit state along \(z\) and corresponding back-action, and \(Q_m\) encoding the other component of the back-action, which is parallel to \(\hat{z} \times \vec{S}_i\). The measurement outcomes are Gaussian distributed, with \(\bar{I}_m^2 + \bar{Q}_m^2 = \bar{n}\nu T_m\) (see text). (B) Schematic of experiment mounted to the base plate of a dilution refrigerator. Readout pulses are transmitted through the strongly coupled port of the resonator, via an isolator and circulator, to the signal port (Sig) of a JPC. The idler port (Idl) is terminated in a 50 Ω load. The amplified signal output is routed via the circulator and further isolators (not shown) to a High Electron Mobility Transistor (HEMT) amplifier operated at 4 K, and subsequently demodulated and digitized at room temperature. (C) False color photograph of the transmon qubit in compact resonator with qubit and resonator parameters. Inset is a scanning electron micrograph showing the Al/AlO\(_x\)/Al junction-based SQUID loop at the center of the transmon.
n = 5
qubit cavity
A
8 ms
Rx(q)
Counts
300 x 10
3
q = 0
q = π
q = π/2
2 4 6 8
Tm = 240 ns
q = π
Im / s
-4
0
4
300 x 10^3
Counts
T_m = 240 ns
θ = 0
θ = π/2
θ = π
**Fig. 2.** (A) Pulse sequence for strong measurement. An initial qubit rotation \( R_x(\theta) \) of \( \theta \) radians about the x-axis is followed by an 8 \( \mu \)s readout pulse with drive power such that \( \bar{n} = 5 \). (B) Individual measurement records. The data are smoothed with a binomial filter with a \( T_m = 240 \) ns time constant, and scaled by the experimentally determined standard deviation (\( \sigma \)). Black dotted lines indicate \( 4\sigma \) deviation events. The qubit is initially measured to be in the excited state, and quantum jumps between excited and ground states are clearly resolved. The center of the ground and excited state distributions are represented as horizontal dotted lines. (C) Histograms of the initial 240 ns record of the readout pulse along \( I_m \) axis, for \( \theta = 0, \pi/2, \pi \). Finite qubit temperature and \( T_1 \) decay during readout are visible as population in the undesired qubit state.
A

State preparation

Variable strength measurement with outcome \( \langle I_m, Q_m \rangle \)

Tomography

\( R_x(\pi/2) \)
\( R_y(\pi/2) \)
\( R_x(-\pi/2) \)
or Id

Repetitive

\( n = 5 \times 10^{-6} \)
times for each value of \( \pi \)

\( \pi = 5 \)

\( I_m = 240 \text{ ns} \)

\( n = 5 \)

\( I_m / \sigma \)

\( Q_m / \sigma \)

\( n = 5 \times 10^{-3} \)

\( n = 5 \times 10^{-2} \)

\( n = 5 \times 10^{-1} \)

\( n = 5 \)

\( X = \pm 1 \)
or

\( Y = \pm 1 \)
or

\( Z = \pm 1 \)

B

Color scale:

0 Max

\( -0.1 \)
\( 0 \)
\( 1 \)
Fig. 3. (A) Pulse sequence for quantifying measurement back-action. The measurement strength was varied linearly in amplitude from $\sqrt{n} = 0$ to $\sqrt{5}$. Conditional maps of $\langle X \rangle_c$, $\langle Y \rangle_c$, $\langle Z \rangle_c$ versus measurement outcome $(I_m/\sigma, Q_m/\sigma)$ were constructed using 201 by 201 bins. (B) Results are shown increasing by decades from $\bar{n} = 5 \times 10^{-3}$ to 5. The left column shows a two-dimensional histogram of all scaled measurement outcomes recorded during the variable strength readout pulse. The three rightmost columns are tomograms showing $\langle X \rangle_c$, $\langle Y \rangle_c$, $\langle Z \rangle_c$ versus associated $(I_m/\sigma, Q_m/\sigma)$ bin.
**Fig. 4.** Correlation between back-action and measurement outcome: (A) Experimental data for correlated back-action signal along $z$, $(\partial \langle Z \rangle_c / \partial I_m)\sigma$, and along $x$, $(\partial \langle X \rangle_c / \partial Q_m)\sigma$, evaluated at $(I_m, Q_m) = 0$, are plotted versus $\bar{I}_m/\sigma$. For weak measurement strength, the slopes at the origin (represented by solid and dashed line, for $z$ and $x$, respectively) agree with theoretical predictions including first order corrections for $T_1$ and $T_2$. The solid curve is the full theoretical expression for the $x$ back-action plotted with $\eta = 0.2$, $\tilde{Q}_m = 1.28\bar{I}_m$, and $\exp(-\tau/T_2) = 0.58$. (B) Experimental data for unconditioned $\langle Y \rangle$ versus $\bar{I}_m/\sigma$. The data show the expected measurement-induced dephasing when the measurement outcome is not used to condition the perturbed qubit state. The dephasing rate is proportional to $(\bar{I}_m/\sigma)^2$, resulting in the apparent Gaussian dependence of $\langle Y \rangle$ vs $\bar{I}_m/\sigma$. The theoretical expression for $\langle Y \rangle$ vs $\bar{I}_m/\sigma$ with parameters listed above is shown as a solid curve.
Supplementary Materials for

Quantum back-action of variable strength measurement


Correspondence to: michael.hatridge@yale.edu.

This PDF file includes:

Materials and Methods
Supplementary Text
Figs. S1 to S3

Other Supplementary Materials for this manuscript includes the following:

Movie S1
1 Materials and Methods

1.1 Qubit–resonator characteristics

The compact resonator, with resonant frequency $f_r = 7.545$ GHz, is lithographically defined in a Nb thin film on a sapphire substrate and capacitively coupled to input and output coplanar waveguide transmission lines, with input and output coupling quality factors $Q_{in} \sim 100,000$ and $Q_{out} = 1400$, respectively (Fig. 1C). The transmon qubit, patterned using electron-beam lithography, consists of an Al/AlO$_x$/Al junction-based SQUID loop shunted by an interdigitated Al thin-film capacitor. The $|g\rangle$ to $|e\rangle$ transition frequency of the qubit is $f_q = 5.676$ GHz. The transmon-resonator coupling is designed so that the dispersive shift ($\chi/2\pi = 5.4$ MHz) is nearly equal to the bandwidth ($\kappa/2\pi = f_r\left(1/Q_{in} + 1/Q_{out}\right) \approx f_r/Q_{out} = 5.8 \pm 0.5$ MHz) of the compact resonator. The qubit $T_1$ is 2.8 $\mu$s, and $T_2$, measured by a Ramsey experiment ($T_{2R}$), evolved from 2.0 to 0.7 $\mu$s over the ten cooldowns of the experiment.

1.2 Photon number calibration

The drive power applied on the input of the compact resonator was calibrated via an AC Stark shift experiment (3) to find the average number of photons ($\bar{n}$) inside the resonator when a measurement pulse traverses it. The qubit frequency was recorded while a continuous microwave tone was applied at the drive frequency $f_d = 7.542$ GHz (centered between $f_q$ and $f_e = f_q - \chi$). As shown in Fig. S1, the qubit frequency decreases with increasing drive power, with a slope of 170 kHz/µW. The dispersive shift was separately determined by spectroscopy to be $\chi/2\pi = 5.4 \pm 0.5$ MHz. Since an average of a single photon at the drive frequency shifts the qubit frequency by $\chi$, we calculate that the drive power of 33 $\mu$W at the generator results in $\bar{n} = 1$ inside the resonator. Using this estimate, we find the average power circulating in the resonator, $P = \bar{n}hf_r\kappa$, which is plotted on the top axis of Fig. S1. The ratio of the applied drive power to the circulating power is approximately 110 dB, which is consistent with that expected from the attenuation installed on the input line and the input quality factor $Q_{IN} \sim 100,000$. This calibration was used to indicate the drive power of measurement pulses throughout the text by specifying the resulting $\bar{n}$ inside the resonator.

1.3 JPC characteristics and measurement efficiency calibration

The JPC was fabricated with a shunted Josephson ring modulator in order to achieve the necessary tunability to match the frequencies of the amplifier and compact resonator (24). It was operated with a power gain $G = 12.5$ dB and bandwidth of 6 MHz. The $-1$ dB saturation power of the amplifier was found to be 1–2 photons referred to the output of the qubit resonator (see sec. 1.2 for photon number calibration). The system noise power was measured using a signal analyzer with a noise increase of 3.5 dB observed when the amplifier was on versus off. Assuming quantum-limited performance for the JPC and negligible losses between the qubit
resonator and amplifier, this places a bound on the quantum efficiency of the measurement chain of \( \eta \leq 0.55 \pm 0.03 \).

The calibration of \( \tilde{n} \) in sec. 1.2 provides an alternate estimate for the measurement efficiency \( \eta \), since it is related to the apparent measurement strength \( I_m/\sigma \). The apparent measurement strength is given by

\[
\tilde{I}_m/\sigma = \sqrt{2\tilde{n}\eta N_m \sin(\vartheta/2)},
\]

where \( \vartheta = 75 \pm 1^\circ \) for our experiment (Fig. 1B) and \( N_m \) is the number of elementary measurements performed. The value of \( \vartheta \) is determined directly from measurement histograms more accurately than either \( \chi \) or \( \kappa \). Since we record the average of \( I(t) \) and \( Q(t) \) over a measurement time \( T_m \), we can write \( N_m \sim BT_m \) where \( B \) is the bandwidth of the measurement system. The resonator bandwidth \( \kappa \) is an upper bound on \( B \), though \( B \) may be reduced by the bandwidth of the JPC and the rest of the measurement chain. For \( T_m = 240 \) ns, we estimate \( N_m \) to range from 4 to 9. Using the data in Fig. 3 we find the apparent measurement strength \( \tilde{I}_m/\sigma = 2.4 \) at \( \tilde{n} = 5 \), and therefore estimate \( \eta \) to range from 0.2 to 0.4. This estimate is in reasonable agreement with \( \eta \leq 0.55 \) calculated from the noise rise experiment presented above.

### 1.4 Calculating \( \frac{\partial \langle Z \rangle_c}{\partial \tilde{I}_m} \) and \( \frac{\partial \langle X \rangle_c}{\partial Q_m} \) versus \( \tilde{I}_m/\sigma \)

In order to produce the back-action data in Fig. 4A, the slopes of \( \langle Z \rangle_c \) and \( \langle X \rangle_c \) were extracted by fitting to data along the \( Q_m = 0 \) and \( I_m = 0 \) bins, respectively (see Fig. S2 for example curves and respective fits). In accordance with the form of \( z_f \) given in Eq. (12), the data were fit to a hyperbolic tangent function. The data were weighted by the number of counts recorded in each bin to take into account the uncertainty due to variable counts per bin. The slope at \( I_m = 0 \) was calculated from the fit coefficients. This routine is robust, and fails only when \( z_f \) changes rapidly compared to the bin spacing. For \( x_f \) given in Eq. (12), the full form is in general more complicated to fit, as its sinusoidal dependence on \( Q_m \) is also suppressed by a factor \( \exp \left( -\tilde{I}_m^2/\sigma^2 (1 - \eta) \right) \). Thus, at large values of \( I_m/\sigma \), where the sinusoidal dependence of \( x_f \) on \( Q_m \) would become relevant, \( x_f \) also goes unconditionally to zero. This fit is further complicated by \( T_2 \) effects which additionally suppress \( x_f \). Consequently, \( x_f \) was fit with a linear dependence, with the fit slope becoming less reliable at higher \( I_m/\sigma \).

The effective measurement strength \( \tilde{I}_m/\sigma \) was separately extracted from measurement histograms. The entire experimental protocol shown in Fig. 3 was replicated with the first qubit rotation of \( R_x(0) \) and \( R_x(\pi) \), which together with preparation of the ground state by post-selection prepare \( |g \rangle \) and \( |e \rangle \), respectively. The most probable \( (I_m, Q_m) \) outcome was extracted with a peak finding algorithm and identified as \( (\tilde{I}_m^g, \tilde{Q}_m) \), for the ground and excited states. Values for \( \tilde{I}_m^g,e \) were found to be symmetric about the \( Q_m = 0 \) axis, and the \( \tilde{I}_m/\sigma = (\tilde{I}_m^e - \tilde{I}_m^g)/(2\sigma) \) was plotted as the horizontal coordinate in Fig. 4.
1.5 Purity of the qubit state after an imperfect measurement

The purity of the qubit state after measurement can be expressed by the length of its Bloch vector, which should remain unity after a perfect measurement if the state was previously known. As shown in Eq. (13), in the case of an imperfect measurement ($\eta < 1$), this length is in general less than 1. However, the purity depends on the measurement outcome; sufficiently large $|I_m|$ outcomes correspond to purity approaching unity, while the purity is reduced for $I_m \sim 0$. This region of reduced purity is a general feature arising from the added classical uncertainty in an imperfect measurement. As the apparent measurement strength increases, the region of low purity is reduced and these outcomes become less probable. This behavior is verified experimentally using the length of the Bloch vector squared plotted in Fig. S3A. The graph also shows the effect of $T_1$ as an overall decrease of purity for outcomes corresponding to $I_m \gg 0$. This asymmetry arises in our data because the relaxation rate ($\Gamma \downarrow$) is a factor of ten higher than the excitation rate ($\Gamma \uparrow$). Therefore, a qubit in the excited state after the middle measurement can relax before the final measurement, whereas a qubit in the ground state remains essentially stationary.

We address the effect of $T_1$ and $T_2$ to first order by modifying Eq. (13), resulting in

$$R^2 = \left( x_f^2 + y_f^2 \right) e^{-2\tau/T_2} + \left( z_f + 1 \right) e^{-\tau/T_1} - 1$$

In Fig. S3B, we plot this modified theory for measurement strengths corresponding to the data in part A which show good qualitative agreement between theory and experiment.

2 Supplementary Text

2.1 Theoretical analysis

We begin the description of the measurement process by considering the propagating mode associated with the measurement pulse. The measurement pulse is assumed to have a temporal extent $T_m$, such that the pulse bandwidth $1/T_m$ is small compared to the bandwidth of the resonator and the JPC. The field quadratures of this mode are defined as $\hat{I} = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{Q} = (\hat{a} - \hat{a}^\dagger)/i$. The average photon occupancy for a general coherent state of the mode, $|\alpha\rangle$ is $\langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2$, while with this definition, the variance of each field quadrature is $\sigma_0^2 = \langle \alpha | (\hat{I} - \langle \hat{I} \rangle)^2 | \alpha \rangle = \langle \alpha | (\hat{Q} - \langle \hat{Q} \rangle)^2 | \alpha \rangle = 1/4$. We can rewrite the field quadratures as

$$\frac{\hat{I}}{\sigma_0} = \hat{a} + \hat{a}^\dagger \quad \text{and} \quad \frac{\hat{Q}}{\sigma_0} = \frac{\hat{a} - \hat{a}^\dagger}{i}.$$  

Later in our derivation, these expressions will clarify the increase in variance due to both phase-preserving amplification and measurement inefficiency. Note that we can write the commutation relation between $\hat{I}$ and $\hat{Q}$ as $[\hat{I}, \hat{Q}] = 2i\sigma_0^2$. 

A note on the organization of this supplement: before discussing the measurement process as a whole in sec. 2.1.2, we first describe the crucial properties of the JPC in sec. 2.1.1.

2.1.1 Input-output relations of the JPC

An ideal, quantum-limited, phase-preserving amplifier like the JPC can be described as a two port device (called signal and idler ports in our case), whose scattering properties are determined by a drive applied to a third (pump) port (21). The input-output scattering relations for an ideal JPC processing single-tone microwave pulses of duration much longer than its inverse bandwidth can be written as

\[ \hat{a}_{\text{sig}}^{\text{out}} = \sqrt{G} \hat{a}_{\text{sig}}^{\text{in}} + i \sqrt{G - 1} \hat{a}_{\text{idl}}^{\dagger \text{in}}, \]
\[ \hat{a}_{\text{idl}}^{\text{out}} = \sqrt{G} \hat{a}_{\text{idl}}^{\text{in}} + i \sqrt{G - 1} \hat{a}_{\text{sig}}^{\dagger \text{in}}, \tag{4} \]

where \( \hat{a}_{\text{sig}}^{\text{in/out}} \) and \( \hat{a}_{\text{idl}}^{\text{in/out}} \) are the incoming and outgoing field mode operators for the signal (Sig) and idler (Idl) ports and \( G \) is the pump-dependent amplifier power gain.

As shown in Fig. 1B, after further amplification through a HEMT amplifier, the output of the signal port is sent to a mixer allowing us to record both output quadratures. Therefore, using Eqs. (3) and (4) to write the output of the of the JPC signal mode referred to its input, we find that we ultimately measure the two physical observable,

\[ \hat{I}_m = \frac{1}{\sqrt{G}} \hat{I}_{\text{sig}}^{\text{out}} = \hat{I}_{\text{sig}}^{\text{in}} + \frac{\sqrt{G - 1}}{\sqrt{G}} \hat{Q}_{\text{idl}}^{\text{in}}, \]
\[ \hat{Q}_m = \frac{1}{\sqrt{G}} \hat{Q}_{\text{sig}}^{\text{out}} = \hat{Q}_{\text{sig}}^{\text{in}} + \frac{\sqrt{G - 1}}{\sqrt{G}} \hat{I}_{\text{idl}}^{\text{in}}. \tag{5} \]

In general, these two observables do not commute and therefore they cannot be measured simultaneously with arbitrary precision. However, in the limit of high gain \( G \gg 1 \), the two observables converge to \( \hat{I}_m = \hat{I}_{\text{sig}}^{\text{in}} + \hat{Q}_{\text{idl}}^{\text{in}} \) and \( \hat{Q}_m = \hat{Q}_{\text{sig}}^{\text{in}} + \hat{I}_{\text{idl}}^{\text{in}} \), which do commute, and form a complete set of commuting observables on \( \mathcal{H}_{\text{sig}} \otimes \mathcal{H}_{\text{idl}} \), the joint Hilbert space of the signal and idler modes (see sec. 2.1.3 for details of the derivation).

2.1.2 Back-action on the qubit state after a phase-preserving measurement of the resonator

To derive the back-action on the qubit due to the measurement of a qubit-resonator entangled state, we assume that the state \( c_g |g\rangle \otimes |\alpha_g\rangle + c_e |e\rangle \otimes |\alpha_e\rangle \) is input to the signal port of the JPC which we suppose here to be ideal (inefficiency of the following measurement chain is dealt with later). As shown in sec. 2.1.1, the amplifier’s input state is actually a combination of both signal and idler inputs, resulting in a modified pointer state \( |\alpha_s, \alpha_i\rangle \). When the idler has no input signal, only quantum fluctuations are present, and the full state of the system can be written as \( c_g |g\rangle \otimes |\alpha_g, 0\rangle + c_e |e\rangle \otimes |\alpha_e, 0\rangle \), where 0 represents vacuum input to the idler.
As shown in Fig. 1A, the pointer states $|\alpha_g\rangle$ and $|\alpha_e\rangle$ can be described by their mean outcomes $\alpha_g = -\bar{I}_m + i\bar{Q}_m$, $\alpha_e = \bar{I}_m + i\bar{Q}_m$, where we have assumed that the drive frequency of the measurement pulse is centered between $f_p^g$ and $f_p^e = f_p^g - \chi$ as in the experiment. The mean outcomes $\bar{I}_m$ and $\bar{Q}_m$ are related to other experimental parameters by $\bar{I}_m^2 + \bar{Q}_m^2 = |\alpha|^2 = \bar{n}\kappa T_m$ and $\bar{I}_m/\bar{Q}_m = \tan(\vartheta/2) = \chi/\kappa$. We perform a measurement of the observables $(\bar{I}_m, \bar{Q}_m)$ and record the outcome $(I_m, Q_m)$, in the process projecting the joint state of the signal and idler modes on to a unique eigenstate $|\psi_{I_mQ_m}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_I$. The new combined state of the qubit, signal and idler modes after projection is $c_g \langle \psi_{I_mQ_m} | \alpha_g, 0 \rangle |g\rangle + c_e \langle \psi_{I_mQ_m} | \alpha_e, 0 \rangle |e\rangle \otimes |\psi_{I_mQ_m}\rangle$. The back-action of the measurement on the qubit state can be summarized as follows. An initial state of the qubit, represented by initial density matrix $\rho_i$, is conditionally transformed after measurement with outcome $(I_m, Q_m)$ to

$$\rho_i \xrightarrow{\text{msmt}} \rho_f(I_m, Q_m) = \frac{M_{I_mQ_m}\rho_i M_{I_mQ_m}^\dagger}{\text{Tr}(M_{I_mQ_m}\rho_i M_{I_mQ_m}^\dagger)}, \quad (6)$$

with a probability distribution for the measurement outcome $(I_m, Q_m)$ given by $P(I_m, Q_m) = \text{Tr}(M_{I_mQ_m}\rho_i M_{I_mQ_m}^\dagger)$ (2).

The Kraus operator, $M_{I_mQ_m}$, calculated in sec. 2.1.4, is found to be

$$M_{I_mQ_m} = \frac{1}{\sqrt{\pi}} e^{-(Q_m - \bar{Q}_m)^2/4\sigma^2_m} \begin{pmatrix} e^{-(I_m + \bar{I}_m)^2/4\sigma^2_m} & \frac{iI_mQ_m}{2\sigma^2_m} & 0 & e^{-\bar{I}_mQ_m/2\sigma^2_m} \\ 0 & e^{-(I_m - \bar{I}_m)^2/4\sigma^2_m} & e^{-\bar{I}_mQ_m/2\sigma^2_m} & e^{-I_mQ_m/2\sigma^2_m} \end{pmatrix}, \quad (7)$$

where $\sigma^2_m = 1/2$.

Let us now assume that the initial state $\rho_i$ is the state polarized along the Bloch sphere $y$-axis so that $(x_i, y_i, z_i) = (0, 1, 0)$. The probability distribution for the outcome $(I_m, Q_m)$ is given by

$$P(I_m, Q_m) = \frac{1}{8\pi\sigma^2_m} \exp\left(-\frac{(Q_m - \bar{Q}_m)^2}{2\sigma^2_m}\right) \left[ \exp\left(-\frac{(I_m - \bar{I}_m)^2}{2\sigma^2_m}\right) + \exp\left(-\frac{(I_m + \bar{I}_m)^2}{2\sigma^2_m}\right) \right]. \quad (8)$$

Firstly, $P$ is found to be an equal superposition of two normal distributions centered on $(-\bar{I}_m, \bar{Q}_m)$ and $(\bar{I}_m, \bar{Q}_m)$ as expected since the qubit was in an equal superposition of $|g\rangle$ and $|e\rangle$. Secondly, the variance of the outcomes in each quadrature is given by $\sigma^2_m = 1/2$. Since the variance of each quadrature of the coherent state $|\alpha\rangle$ input to the signal port is given by $\sigma^2_0 = \langle \alpha | (\hat{I}_{\text{Sig}} - \langle \hat{I}_{\text{Sig}} \rangle)^2 | \alpha \rangle = \langle \alpha | (\hat{Q}_{\text{Sig}} - \langle \hat{Q}_{\text{Sig}} \rangle)^2 | \alpha \rangle = 1/4$, we find that $\sigma^2_m = 2\sigma^2_0$. Thus, as expected, phase-preserving amplification adds an extra half photon of noise which can be physically identified to originate from the quantum noise input to the idler mode.

Next, using Eq. (6), we calculate the final density matrix $\rho_f$ and find that the measurement
back-action corresponds to a final qubit Bloch vector:

\[ x_f(I_m, Q_m) = \text{sech} \left( \frac{I_m I_m}{\sigma_m^2} \right) \sin \left( \frac{Q_m I_m}{\sigma_m^2} \right), \]
\[ y_f(I_m, Q_m) = \text{sech} \left( \frac{I_m I_m}{\sigma_m^2} \right) \cos \left( \frac{Q_m I_m}{\sigma_m^2} \right), \]
\[ z_f(I_m) = \tanh \left( \frac{I_m I_m}{\sigma_m^2} \right). \tag{9} \]

We see from Eq. (9), that \( x_f, y_f, z_f \) can each take a continuum of values between \(-1\) and \(1\) as expected for a general finite-strength measurement. Further, since \( x_f^2 + y_f^2 + z_f^2 = 1 \), the qubit remains in a pure state after the measurement despite the extra half photon of quantum noise added by the idler mode. This additional quantum noise forces the qubit to diffuse in all directions on the 2-D surface of the Bloch sphere. However, as Eq. (9) shows, recording only the unpredictable outcome \((I_m, Q_m)\) of a perfect measurement provides sufficient information for full knowledge of the new state of the qubit after measurement. This principle that the unpredictable outcome of a perfect measurement provides sufficient information for full knowledge of the back-action is naturally also true for the case of noiseless phase-sensitive amplification, though the details of the measurement back-action differ \((8)\).

The effect of imperfections in the measurement chain arising from losses after the first amplification provided by the JPC is taken into account by adding extra classical noise over the fundamental quantum noise. Let \( \sigma^2 \) be the observed variance after an imperfect measurement. We define a measurement efficiency \( \eta = \sigma_m^2/\sigma^2 \) which by definition is no greater than unity. This efficiency is linked to the inverse of Caves’ added noise number \( A \) through the relation \( 1/\eta = A + 1/2 \), with \( A \geq 1/2 \) for quantum-limited, phase-preserving amplification \((19)\). This extra added classical noise results in a modification to the back-action in the following manner

\[ (x, y, z)_f(I_m, Q_m) = \int \int dI dQ \mathbb{P}(I, Q | I_m, Q_m) (x, y, z)_f(I, Q), \tag{10} \]

where \( \mathbb{P}(I, Q | I_m, Q_m) \) denotes the conditional probability that a perfect measurement would give the outcome \((I, Q)\) while our imperfect one has given \((I_m, Q_m)\). \( \mathbb{P}(I, Q | I_m, Q_m) \) is calculated in the sec. 2.1.5 to be

\[ \mathbb{P}(I, Q | I_m, Q_m) = \frac{1}{2\pi \eta (1-\eta) \sigma^2} e^{-\frac{(Q-Q_m)(1-\eta)Q_m}{2(1-\eta)\sigma^2}} e^{-\frac{(I-I_m)^2}{2\sigma_m^2} + e^{-\frac{(I+I_m)^2}{2\sigma_m^2}} \cdot \left( e^{-\frac{(I-I_m)^2}{2\sigma^2}} + e^{-\frac{(I+I_m)^2}{2\sigma^2}} \right). \tag{11} \]

Applying equation (10), we have that the final qubit state conditioned on the measurement
outcome \((I_m, Q_m)\), expressed as a final qubit Bloch vector \(\mathbf{S}_f = (x_f, y_f, z_f)\), is

\[
x_f^\eta(I_m, Q_m) = \text{sech}\left(\frac{I_m \bar{I}_m}{\sigma^2}\right) \sin\left(\frac{Q_m \bar{I}_m}{\sigma^2} + \frac{\bar{Q}_m I_m}{\sigma^2} \left(\frac{1 - \eta}{\eta}\right)\right) e^{-\frac{I_m^2}{\sigma^2} \left(\frac{1 - \eta}{\eta}\right)},
\]

\[
y_f^\eta(I_m, Q_m) = \text{sech}\left(\frac{I_m \bar{I}_m}{\sigma^2}\right) \cos\left(\frac{Q_m \bar{I}_m}{\sigma^2} + \frac{\bar{Q}_m I_m}{\sigma^2} \left(\frac{1 - \eta}{\eta}\right)\right) e^{-\frac{I_m^2}{\sigma^2} \left(\frac{1 - \eta}{\eta}\right)},
\]

\[
z_f^\eta(I_m) = \tanh\left(\frac{I_m \bar{I}_m}{\sigma^2}\right).
\]

The length of the final Bloch vector squared, given by

\[
R^2 = x_f^2 + y_f^2 + z_f^2 = 1 - \text{sech}^2\left(\frac{I_m \bar{I}_m}{\sigma^2}\right) \left(1 - \exp\left(-\frac{2I_m^2}{\sigma^2} \frac{1 - \eta}{\eta}\right)\right),
\]

will be in general less than 1 and therefore the qubit almost always fails to end in a pure state for \(\eta < 1\). Here the purity of the state after measurement is directly linked to the measurement outcome, with large amplitude \(I_m\) outcomes purer than those near zero.

The parameter \(\bar{I}_m/\sigma\) can be identified as the apparent measurement strength since the measurement becomes more strongly projective as \(\bar{I}_m/\sigma\) increases. To see this, note that \(I_m\) outcomes in the center of the distribution (\(|I_m| \sim 0\)) become less probable, since, as shown from Eq. (8), the two Gaussian distributions centered on \(-\bar{I}_m\) and \(\bar{I}_m\) move further apart. Additionally, most \(I_m\) outcomes correspond to a final Bloch vector \(\mathbf{S}_f\) that is projected onto the \(z\)-axis. When \(\bar{I}_m/\sigma \to \infty\), we recover the situation of infinite-strength measurement where the only possible final qubit states are the poles of the Bloch sphere. Note that this strength depends both on the strength of the readout and on the amplifier efficiency.

For weak apparent measurement strength \((\bar{I}_m/\sigma, \bar{Q}_m/\sigma \ll 1)\), we can approximate Eq. (12) to first order in \((I_m/\sigma, Q_m/\sigma)\) as

\[
x_f^\eta(I_m, Q_m) \simeq \left(\frac{\bar{I}_m}{\sigma}\right) \frac{Q_m}{\sigma},
\]

\[
y_f^\eta(I_m, Q_m) \simeq 1,
\]

\[
z_f^\eta(I_m, Q_m) \simeq \left(\frac{\bar{I}_m}{\sigma}\right) \frac{I_m}{\sigma}.
\]

This form shows that the back-action on the \(x\)- and \(z\)- component of the qubit state after measurement directly correspond to the \(I\)- and \(Q\)- component of the measurement outcome, respectively. The amplitude of the measurement back-action is symmetric in \(x\) and \(z\), and determined by the apparent measurement strength \(\bar{I}_m/\sigma\).
Finally, we find that the correlation of the back-action with the measurement outcome is
\[
\left. \sigma \frac{\partial x^n_f}{\partial Q_m} \right|_{(m,Q_m)=(0,0)} = \frac{I_m}{\sigma} \cos \left( \frac{Q_m I_m}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right) \right) e^{-\frac{I_m^2}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right)},
\]
\[
\left. \sigma \frac{\partial y^n_f}{\partial Q_m} \right|_{(m,Q_m)=(0,0)} = -\frac{I_m}{\sigma} \sin \left( \frac{Q_m I_m}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right) \right) e^{-\frac{I_m^2}{\sigma^2} \left( \frac{1 - \eta}{\eta} \right)},
\]
which to first order in \( I_m/\sigma, Q_m/\sigma \) reduces to
\[
\left. \sigma \frac{\partial x^n_f}{\partial Q_m} \right|_{(m,Q_m)=(0,0)} \approx \frac{I_m}{\sigma},
\]
\[
\left. \sigma \frac{\partial y^n_f}{\partial Q_m} \right|_{(m,Q_m)=(0,0)} \approx 0,
\]
\[
\left. \sigma \frac{\partial z^n_f}{\partial I_m} \right|_{I_m=0} \approx \frac{I_m}{\sigma}.
\]

### 2.1.3 Showing \( \hat{I}_m \) and \( \hat{Q}_m \) form a complete set of commuting observables

We now prove that the two observables \( \hat{I}_m = \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}} \) and \( \hat{Q}_m = \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}} \) form a complete set of commuting observables on \( \mathcal{H}_{\text{Sig}} \otimes \mathcal{H}_{\text{Idl}} \), the joint Hilbert space of the signal and idler modes of the JPC. This result implies that a simultaneous measurement of these two observables projects the joint state of the signal and idler modes to a unique common eigenstate of both observables. First, the two observables commute since trivially
\[
[\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}}] = [\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}] - [\hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}] = 0.
\]
Next, in order to prove completeness, let us assume that there exists another observable \( \hat{O} \) on \( \mathcal{H}_S \otimes \mathcal{H}_I \) such that
\[
[\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{O}] = 0 \quad \text{and} \quad [\hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}}, \hat{O}] = 0. \tag{17}
\]
We will show that \( \hat{O} \) is necessarily of the form \( F(\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}}) \), which does not depend on \( \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Idl}} \) and \( \hat{Q}_{\text{Idl}} - \hat{I}_{\text{Sig}} \). To begin, \( \hat{O} \) can in general be written as a function \( f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}) \) of all four quadratures. We have,
\[
[\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{O}] = [\hat{I}_{\text{Sig}}, f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}})] + [\hat{Q}_{\text{Idl}}, f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}})]
\]
\[
= i\hbar \frac{\partial}{\partial Q_{\text{Sig}}} f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}) - i\hbar \frac{\partial}{\partial I_{\text{Idl}}} f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}})
\]
\[
= i\hbar \left( \frac{\partial}{\partial Q_{\text{Sig}} - \hat{I}_{\text{Idl}}} f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}). \right.
\]
Together with Eq. (17), this implies
\[
\frac{\partial}{\partial (\hat{Q}_{\text{Sig}} - \hat{I}_{\text{Idl}})} f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}) = 0.
\]

Similarly, one finds
\[
\frac{\partial}{\partial (\hat{Q}_{\text{Idl}} - \hat{I}_{\text{Sig}})} f(\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}, \hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}) = 0.
\]
Therefore \( f \) does not depend on \( \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Idl}} \) and \( \hat{Q}_{\text{Idl}} - \hat{I}_{\text{Sig}} \). Therefore \( f \) is only function of \( \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}} \) and \( \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}} \), i.e. \( \hat{O} = F(\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}}) \). Hence we have proven that \( \hat{I}_m = \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}} \) and \( \hat{Q}_m = \hat{Q}_{\text{Sig}} + \hat{I}_{\text{Idl}} \) form a complete set of commuting observables. This is the mathematical representation of the physical intuition that the output of a high gain amplifier is ‘classical’. The ‘added noise’ of the amplifier accounts for the fact that, for vacuum input, while the signal and idler input ports obey \( \langle 0 | \hat{Q}^2 | 0 \rangle = \langle 0 | \hat{I}^2 | 0 \rangle = 1/4 \), the signal output port obeys \( \langle 0 | \hat{Q}_m^2 | 0 \rangle = \langle 0 | \hat{I}_m^2 | 0 \rangle = 1/2 \).

### 2.1.4 Calculation of Kraus operator \( M_{I_m,Q_m} \)

The Kraus operator \( M_{I_m,Q_m} \) (2) is defined as
\[
M_{I_m,Q_m} = \left( \begin{array}{cc} \langle \psi_{I_m,Q_m} | \alpha_g, 0 \rangle & 0 \\ 0 & \langle \psi_{I_m,Q_m} | \alpha_e, 0 \rangle \end{array} \right) \tag{18}
\]

Let us now calculate \( \xi_\alpha(I_m, Q_m) = \langle \psi_{I_m,Q_m} | \alpha, 0 \rangle \) needed to find \( M_{I_m,Q_m} \). The following identity follows from the definition of a coherent state
\[
\langle \psi_{I_m,Q_m} | \hat{a}_{\text{Sig}} - i\hat{a}_{\text{Idl}} | \alpha, 0 \rangle = \alpha \langle \psi_{I_m,Q_m} | \alpha, 0 \rangle = \alpha \xi_\alpha(I_m, Q_m). \tag{19}
\]

However, we also have
\[
\langle \psi_{I_m,Q_m} | \hat{a}_{\text{Sig}} - i\hat{a}_{\text{Idl}} | \alpha, 0 \rangle = \langle \psi_{I_m,Q_m} | \frac{\hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}}{2\sigma_0} | \alpha, 0 \rangle + i \langle \psi_{I_m,Q_m} | \frac{\hat{Q}_{\text{Sig}} - \hat{I}_{\text{Idl}}}{2\sigma_0} | \alpha, 0 \rangle. \tag{20}
\]

As \( |\psi_{I_m,Q_m}\rangle \) is an eigenstate of \( \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}} \) with eigenvalue \( I_m \), we have
\[
\langle \psi_{I_m,Q_m} | \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}} | \alpha, 0 \rangle = I_m \langle \psi_{I_m,Q_m} | \alpha, 0 \rangle = I_m \xi_\alpha(I_m, Q_m). \tag{21}
\]

Now note that by the canonical commutation relation \( [\hat{I}_{\text{Sig}}, \hat{Q}_{\text{Sig}}] = [\hat{I}_{\text{Idl}}, \hat{Q}_{\text{Idl}}] = 2i\sigma_0^2 \), we have
\[
\langle \psi_{I_m,Q_m} | \left[ \hat{I}_{\text{Sig}} + \hat{Q}_{\text{Idl}}, \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Idl}} \right] |\psi_{I_m,Q_m}\rangle = 4i\sigma_0^2 \langle \psi_{I_m,Q_m} | \psi_{I_m,Q_m} \rangle = 4i\sigma_0^2 \delta(I_m - \bar{I}_m) \delta(Q_m - \bar{Q}_m), \tag{22}
\]
where we have the fact that the normalized eigenstates $|\psi_{I_m,\tilde{Q}_m}\rangle$ and $|\psi_{I_m,Q_m}\rangle$ are orthogonal except if $\tilde{I}_m = I_m$ and $\tilde{Q}_m = Q_m$. Furthermore, using the fact that $\psi_{I_m,\tilde{Q}_m}$ and $\psi_{I_m,Q_m}$ are eigenstates of $\hat{I}_{\text{Sig}} + \hat{I}_{\text{Ild}}$, we have that

$$
\langle \psi_{I_m,\tilde{Q}_m} | \left[ \hat{I}_{\text{Sig}} + \hat{I}_{\text{Ild}}, \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Ild}} \right] | \psi_{I_m,\tilde{Q}_m} \rangle = (I_m - \tilde{I}_m) \langle \psi_{I_m,\tilde{Q}_m} | \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Ild}} | \psi_{I_m,\tilde{Q}_m} \rangle.
$$

(23)

By Eq. (22) and (23) we have

$$
\langle \psi_{I_m,\tilde{Q}_m} | \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Ild}} | \psi_{I_m,\tilde{Q}_m} \rangle = 4i\sigma_0 \delta(I_m - \tilde{I}_m) \delta(Q_m - \tilde{Q}_m) = -4i\sigma_0 \delta(Q_m - \tilde{Q}_m) \frac{\partial}{\partial I_m} \delta(I_m - \tilde{I}_m).
$$

(24)

Therefore, we have

$$
\langle \psi_{I_m,\tilde{Q}_m} | \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Ild}} | \psi_{I_m,\tilde{Q}_m} \rangle |_{\alpha,0} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dQ_m d\tilde{Q}_m \langle \psi_{I_m,\tilde{Q}_m} | \hat{Q}_{\text{Sig}} - \hat{I}_{\text{Ild}} | \psi_{I_m,\tilde{Q}_m} \rangle \langle \psi_{I_m,\tilde{Q}_m} | \psi_{I_m,\tilde{Q}_m} \rangle |_{\alpha,0}
$$

$$
= -4i\sigma_0 \frac{\partial \xi_\alpha}{\partial I_m} |_{I_m,Q_m}.
$$

(25)

Equations (19), (20), (21) and (25) lead to the following partial differential equation:

$$
\frac{\partial}{\partial I_m} \xi_\alpha(I_m, Q_m) = \frac{(\alpha - I_m)}{4\sigma_0^2} \xi_\alpha(I_m, Q_m).
$$

(26)

In the same way, by considering $\langle \psi_{I_m,\tilde{Q}_m} | \tilde{a}_{\text{Ild}} - i\tilde{a}_{\text{Sig}} | \alpha,0 \rangle$, we have

$$
\frac{\partial}{\partial Q_m} \xi_\alpha(I_m, Q_m) = \frac{(-i\alpha - Q_m)}{4\sigma_0^2} \xi_\alpha(I_m, Q_m).
$$

(27)

The general solution to the system of equations (26) and (27) is given by

$$
\xi_\alpha(I_m, Q_m) = N_\alpha \exp \left( -\frac{1}{2} \frac{(I_m - \alpha)^2}{(2\sigma_0)^2} \right) \exp \left( -\frac{1}{2} \frac{(Q_m + i\alpha)^2}{(2\sigma_0)^2} \right),
$$

(28)

where $N_\alpha$ is a constant independent of $I_m$ and $Q_m$. Using the fact that $\int dI_m dQ_m |\xi_\alpha(I_m, Q_m)|^2 = 1$, we can show that $N_\alpha = \pi^{-1/2} e^{-|\alpha|^2/(2\sigma_0)^2} e^{i\zeta_\alpha}$, where $\zeta_\alpha$ is a phase term. One can further show that this phase is independent of $\alpha$ by noting that

$$
\langle 0,0 | \alpha,0 \rangle = \int dI_m dQ_m \langle 0,0 | \psi_{I_m,\tilde{Q}_m} \rangle \langle \psi_{I_m,\tilde{Q}_m} | \alpha,0 \rangle
$$

$$
= \frac{1}{\pi} e^{-|\alpha|^2/(2\sigma_0)^2} e^{i(\zeta_\alpha - \zeta_0)} \int dI_m e^{-\frac{1}{2} \frac{((I_m - \alpha)^2 + \alpha^2)}{(2\sigma_0)^2}} \int dQ_m e^{-\frac{1}{2} \frac{((Q_m + i\alpha)^2 + Q^2)}{(2\sigma_0)^2}}
$$

$$
= e^{-|\alpha|^2/(2\sigma_0)^2} e^{i(\zeta_\alpha - \zeta_0)}.
$$

(29)
By definition of the coherent state $|\alpha\rangle$, we also know that $\langle 0, 0 | \alpha, 0 \rangle = e^{-|\alpha|^2/2(2\sigma_0)^2}$. Therefore $\zeta_\alpha = \zeta_0$, which indicates that the phase $\zeta_\alpha$ is an arbitrary constant independent of $\alpha$. This completes the calculation of $\xi_\alpha(I_m, Q_m)$.

From the definition of $\xi_\alpha(I_m, Q_m) = \langle \psi_{I_m,Q_m} | \alpha, 0 \rangle$, we can see that $|\xi_\alpha(I_m, Q_m)|^2$ gives the probability density function for recording an outcome $(I_m, Q_m)$ when a coherent state $|\alpha\rangle$ is input at the signal mode of the amplifier and the vacuum state is present on the idler. Using (28) we can show that this implies that $(I_m, Q_m)$ are random variables drawn from a normal distribution with mean $(\text{Re}(\alpha), \text{Im}(\alpha))$ and variance of $2\sigma_0^2 = 1/2$. Thus this derivation shows that phase-preserving amplification increases the variance of a coherent state by a factor of two, or in other words the quantum noise of the idler mode adds an extra half photon worth of noise to the input signal as expected from Caves’ theorem. We therefore define $\sigma_m^2 = 2\sigma_0^2$ to be the variance of a coherent state observed at the output of an ideal, quantum-limited phase-preserving amplifier and can now write the Kraus operator $M_{I_m,Q_m}$ as

$$M_{I_m,Q_m} = \frac{e^{i\zeta_0}}{\sqrt{\pi}} \begin{pmatrix} e^{-|\alpha|^2/4\sigma_m^2} e^{-\langle I_m-\bar{I}_m \rangle^2/2\sigma_m^2} & 0 \\ 0 & e^{-|\alpha|^2/4\sigma_m^2} e^{-\langle Q_m-\bar{Q}_m \rangle^2/2\sigma_m^2} \end{pmatrix}.$$  \hspace{1cm} (30)

We can further simplify $M$ in the context of the current experiment. We take $\alpha_g \equiv -\bar{I}_m + i\bar{Q}_m$ and $\alpha_e \equiv \bar{I}_m + i\bar{Q}_m$, and note that dividing the matrix $M$ by a constant of unit modulus would not change the dynamics of Eq. (6). Therefore,

$$M = 1 e^{-\frac{(Q_m-Q_m)^2}{4\sigma_m^2}} \begin{pmatrix} e^{-\langle I_m+I_m \rangle^2/2\sigma_m^2} & 0 \\ 0 & e^{-\langle Q_m-Q_m \rangle^2/2\sigma_m^2} \end{pmatrix}.$$  \hspace{1cm} (31)

2.1.5 Calculation of $P(I, Q | I_m, Q_m)$

The conditional probability $P(I, Q | I_m, Q_m)$ that a perfect measurement would give outcome $(I, Q)$ while an imperfect one has given $(I_m, Q_m)$ can be calculated through Bayes rule.

$$P(I, Q | I_m, Q_m) = \frac{P(I_m, Q_m | I, Q) P(I, Q)}{\int \int dI dQ P(I_m, Q_m | I, Q) P(I, Q)}$$

where $P(I, Q)$ is the probability of the measurement outcome $(I, Q)$ given in equation (8). Since the extra added classical noise has a variance $\sigma^2 - \sigma_m^2 = (1-\eta)\sigma^2$, we have

$$P(I_m, Q_m | I, Q) = \frac{1}{2\pi(1-\eta)\sigma^2} e^{-\frac{(I_m-I)^2}{2(1-\eta)\sigma^2}} e^{-\frac{(Q_m-Q)^2}{2(1-\eta)\sigma^2}},$$

which implies that

$$\int \int dI dQ P(I_m, Q_m | I, Q) P(I, Q) = \frac{1}{4\pi\sigma^2} e^{-\frac{(Q_m-Q_m)^2}{2\sigma^2}} \left( e^{-\frac{(I_m-I)^2}{2\sigma^2}} + e^{-\frac{(I_m+I)^2}{2\sigma^2}} \right).$$
Therefore $\mathbb{P}(I, Q | I_m, Q_m)$ is

$$
\mathbb{P}(I, Q | I_m, Q_m) = \frac{1}{2\pi \eta (1-\eta) \sigma^2} e^{-\frac{(Q-\eta Q_m-(1-\eta)Q_m)^2}{2\eta (1-\eta) \sigma^2}} \left( \frac{e^{-(I-I_m)^2/2\sigma^2_m}}{e^{-(I-I_m)^2/2\sigma^2} + e^{-(I+I_m)^2/2\sigma^2_m}} + \frac{e^{-(I+I_m)^2/2\sigma^2_m}}{e^{-(I-I_m)^2/2\sigma^2} + e^{-(I+I_m)^2/2\sigma^2_m}} \right).
$$
Fig. S1. Measurement strength calibration. Plot of qubit frequency as a function of applied drive power used to calibrate the rate at which photons leave the resonator for a given drive power, and therefore the measurement strength. Using the slope of 170 kHz/µW and χ/2π = 5.4 MHz, we calculate that a drive power of 33 µW results in an average of 1 photon leaking out of the resonator in a bandwidth κ. The corresponding average circulating power \( P = \bar{n} h f_r \kappa \) is plotted on the top axis.
Fig. S2. (A) Example of $\langle Z \rangle_c$ vs $I_m/\sigma$ data extracted from $Q_m = 0$ line of $\langle Z \rangle_c$ tomogram for three measurement drives. The data shows that the hyperbolic tangent dependence of $\langle Z \rangle_c$ on $I_m/\sigma$ becomes more pronounced as the measurement strength is increased. Linear fits to the data near $I_m = 0$ are shown as dashed lines. (B) Example of $\langle X \rangle_c$ vs $Q_m/\sigma$ data extracted from $I_m = 0$ line of $\langle X \rangle_c$ tomogram for the same drives. The linear dependence of $\langle X \rangle_c$ on $Q_m/\sigma$ for small $\bar{I}_m/\sigma$ is demonstrated by linear fits to the data shown as dashed lines. The suppression of $\langle X \rangle_c$ for large $\bar{I}_m/\sigma$ and imperfect measurement is demonstrated for the strongest drive (green). In this case $\langle X \rangle_c \sim 0$, exponentially suppressing its sinusoidal dependence on $Q_m$. 
Fig. S3. Experimental (A) and theory (B) curves for the length of Bloch vector squared 
$\langle X \rangle^2 + \langle Y \rangle^2 + \langle Z \rangle^2$ as a function of $I_m / \sigma$ for two measurement drives. (A) Data extracted from conditional maps by performing a weighted average over $Q_m$ for each component $\langle X \rangle_c$, $\langle Y \rangle_c$, $\langle Z \rangle_c$ and summing the squares of the results. (B) Theory curves from Eq. (2) for corresponding measurement drives and experimentally determined values of $\bar{I}_m / \sigma$, $T_1$ and $T_2$. The value for $\eta$ was chosen to be 0.55, the maximum value estimated from the noise rise measurement described in the main text.