Abstract

Concurrent Remote Entanglement with Continuous Variables

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Entanglement, a holistic property of compound quantum systems, is one of the quintessential features of quantum mechanics. Generation of entangled states between spatially separated, mutually non-interacting, quantum systems is crucial for large-scale quantum information processing. In particular, concurrent remote entanglement, in which no information is ever directly exchanged between the two quantum systems under consideration, is a key primitive for a scalable, module-based architecture of quantum computing. The use of continuous variables (CV) in concurrent remote entanglement protocols holds the promise of deterministic generation of maximally entangled states. At the same time, it opens the investigation of nonlinear interactions of quantum states of light, which are of fundamental interest in physics.

In this thesis, a new protocol for concurrent remote entanglement with CV of propagating modes of light is proposed. This protocol remotely entangles two distant, stationary, non-interacting qubits by first entangling each of them with a propagating ancilla qubit. In the next step, in contrast to existing protocols, which use a unitary operation to erase the ‘which ancilla is entangled to which stationary qubit’ information, we perform a nonlinear, two-qubit, quantum non-demolition measurement on the ancillas to achieve the same. Subsequently, a single-qubit measurement is performed on each of the ancillas. Depending on the outcomes in these three measurements, the stationary qubits are projected onto a particular entangled Bell-state. For CV implementation, we encode the ancilla qubits in the phase-spaces of propagating temporal modes of light, in particular, in even and odd Schrödinger cat states. After local entanglement generation between each stationary qubit and its associated ancilla, a two-qubit, quantum non-demolition (QND) measurement, XX, on the ancillas is implemented by a nonlinear interaction of the propagating modes with an auxiliary mode, followed by a linear homodyne detection of the latter. The nonlinear interaction is achieved by a novel, Josephson circuit device, the Josephson Parametric Multiplier, which consists of three resonator modes, together with a new, non-linear, four-wave mixing element, the Josephson Four Wave Mixer. In contrast to usual linear scattering devices of Gaussian CV signal processing, this nonlinear device performs non-Gaussian CV signal processing. The final single-qubit measurements
(Z-s) are implemented by homodyne detections. An alternate variation of the above protocol is also proposed. In this case, a two-qubit, QND measurement, ZZ, instead of XX, is performed on the propagating modes. It is done by a joint-photon-number-modulo-2 measurement using an auxiliary transmon qubit. The single-qubit measurements (X-s) are again implemented by linear homodyne detections. Both variations of our protocol generate maximally entangled Bell-states with unit probability in absence of imperfections and can be straightforwardly implemented in superconducting circuit-QED systems.

The inevitable presence of imperfections in realistic quantum systems motivated the search for a CV, concurrent remote entanglement protocol, which is resilient to these inefficiencies. To that end, a modification of the aforementioned protocol is proposed, which incorporates quantum error correction to suppress the decoherence due to photon loss. In this modification, the ancilla qubits are encoded in superpositions of Schrödinger cat states of a given photon-number-parity, a two-qubit, QND ZZ measurement is performed by a joint-photon-number-modulo-4 measurement and the single-qubit measurements are performed by homodyne detections. The resilience of the quantum error-correcting protocol to imperfections is analyzed.
Concurrent Remote Entanglement with Continuous Variables

A Dissertation
Presented to the Faculty of the Graduate School of Yale University in Candidacy for the Degree of Doctor of Philosophy

by
Ananda Roy

Dissertation Directors: Michel Devoret and A. Douglas Stone

December, 2016
Dedicated to the memory of my grandfather
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<td>( \hat{a} )</td>
<td>expectation value of operator ( a ) in rotating frame</td>
</tr>
<tr>
<td>( a, a^\dagger )</td>
<td>photon annihilation/creation operators</td>
</tr>
<tr>
<td>( \tilde{a}, \tilde{a}^\dagger )</td>
<td>photon annihilation/creation operators in rotating frame</td>
</tr>
<tr>
<td>( a^\leftrightarrow(x, t) )</td>
<td>right/left traveling field operator without rotating wave approximation in real space</td>
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<tr>
<td>( a^\leftrightarrow[\omega] )</td>
<td>Fourier transform of traveling field operator without rotating wave approximation</td>
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<td>( a^\leftrightarrow_{\text{RWA}} )</td>
<td>right/left traveling field operator under rotating wave approximation</td>
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<td>( a_{\text{in, out}} )</td>
<td>traveling incoming/outgoing field operator under rotating wave approximation</td>
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<td>( \tilde{a}_{\text{in, out}} )</td>
<td>traveling incoming/outgoing field operator under rotating wave approximation in rotating frame</td>
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<tr>
<td>( A^\leftrightarrow(x, t) )</td>
<td>operators for the amplitude of right/left waves along a transmission line</td>
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<td>( A^\leftrightarrow[\omega] )</td>
<td>Fourier transform of operators for the amplitude of right/left waves along a transmission line</td>
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<td>( A, A^\dagger )</td>
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<td>( \hat{b} )</td>
<td>expectation value of operator ( b ) in rotating frame</td>
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<td>( b, b^\dagger )</td>
<td>photon annihilation/creation operators</td>
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<tr>
<td>( \tilde{b}, \tilde{b}^\dagger )</td>
<td>photon annihilation/creation operators in rotating frame</td>
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<tr>
<td>( b_{\text{in, out}} )</td>
<td>traveling incoming/outgoing field operator under rotating wave approximation</td>
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</table>
\( \tilde{b}_{\text{in,out}} \) traveling incoming/outgoing field operator under rotating wave approximation in rotating frame

\( \mathbf{B}, \mathbf{B}^\dagger \) photon annihilation/creation operators

\( \tilde{c} \) expectation value of operator \( b \) in rotating frame

\( \mathbf{c}, \mathbf{c}^\dagger \) photon annihilation/creation operators

\( \tilde{\mathbf{c}}, \tilde{\mathbf{c}}^\dagger \) photon annihilation/creation operators in rotating frame

\( c_{\text{in,out}} \) traveling incoming/outgoing field operator under rotating wave approximation

\( \tilde{c}_{\text{in,out}} \) traveling incoming/outgoing field operator under rotating wave approximation in rotating frame

\( C \) capacitance

\( |C_{\alpha}^{\pm}\rangle \) even/odd Schrödinger cat state

\( |C_{\alpha}^{\Lambda \mod 4}\rangle \) mod 4 cat state with populations in Fock states \( |4n + \lambda\rangle, \ n \in \mathbb{N} \)

\( C_\ell \) capacitance per unit length of an infinite transmission line

\( \mathbf{d}, \mathbf{d}^\dagger \) photon annihilation/creation operators

\( \mathcal{D}(\alpha) \) displacement operator in phase-space

\( \mathcal{D}(\mathbf{J}) \) Lindblad super-operator for dissipation mediated by the jump-operator \( \mathbf{J} \)

\( E_J \) Josephson energy

\( \mathbf{f}, \mathbf{f}^\dagger \) photon annihilation/creation operators

\( g \) three-wave mixing coefficient

\( \tilde{g} \) dimensionless three-wave mixing coefficient

\( \mathbf{g}, \mathbf{g}^\dagger \) photon annihilation/creation operators

\( g_4 \) four-wave mixing coefficient

\( |g\rangle, |e\rangle \) ground and excited states of a qubit

\( \hbar \) Planck’s constant

\( \mathbf{H} \) Hadamard gate

\( \text{HD}_a \) homodyne detection of mode \( a \)

\( \mathbf{H}_{2\text{ph}} \) two-photon Hamiltonian

\( \mathbf{H}_{2\text{ph}} \) two-photon Hamiltonian in rotating frame

\( \mathbf{H}_{\text{cat–stab}} \) total Hamiltonian for stabilization of even Schrödinger cat state

\( \mathbf{H}_{\text{cross–Kerr}} \) Hamiltonian for the cross-Kerr interaction in the rotating frame
<table>
<thead>
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<td>$H_{\text{mod4}}$</td>
<td>Hamiltonian for the joint-photon-number-modulo-4 measurement</td>
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<td>$H_{\text{ps}}$</td>
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<td>$H_{sr_1 (sr_2)}$</td>
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<td>$I$</td>
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<td>$I_0$</td>
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<td>$J_{2\tilde{n}}$</td>
<td>jump operators connecting the states $</td>
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<td>$L$</td>
<td>inductance</td>
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<td>$L_J$</td>
<td>nonlinear inductance associated with a Josephson junction</td>
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<td>$L_\ell$</td>
<td>inductance per unit length of an infinite transmission line</td>
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<td>$L_{\text{line}}$</td>
<td>Lagrangian of an infinite transmission line</td>
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<td>$\mathcal{L}_{\text{line}}$</td>
<td>Lagrangian density of an infinite transmission line</td>
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<td>$M$</td>
<td>matrix connecting standing modes operators to input mode operators</td>
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<td>$\mathcal{M}$</td>
<td>measurement operator corresponding to homodyne detection</td>
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<td>$n$</td>
<td>photon-number operator</td>
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<td>$n_{a,b}$</td>
<td>photon-numbers of modes $a, b$</td>
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<td>$n_\mu$</td>
<td>photon-number operator of the temporal mode $\mu$</td>
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<td>$N_{\pm}$</td>
<td>normalization of even/odd Schrödinger cat state</td>
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<td>$\tilde{N}_{\pm}$</td>
<td>normalization of even/odd Schrödinger cat state</td>
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<td>$\tilde{N}_\lambda$</td>
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<td>$\bar{\tilde{N}}_\lambda$</td>
<td>normalization of mod 4 cat state $</td>
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<td>p.p.</td>
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<td>$P^p(x_a, x_b)$</td>
<td>probability of outcomes in absence of imperfections</td>
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<tr>
<td>$\bar{P}^p(q_a, q_b)$</td>
<td>probability of outcomes in presence of imperfections</td>
</tr>
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<td>Symbol</td>
<td>Description</td>
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<tr>
<td>$P^\eta_{\text{total}}$</td>
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<td>$P^*$</td>
<td>optimized success probability</td>
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<td>$q$</td>
<td>Fourier transform of flux operator of an infinite transmission line</td>
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<td>$q_{a,b}$</td>
<td>integrated homodyne current for modes $a, b$ in presence of imperfections</td>
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<td>$S$</td>
<td>scattering matrix connecting input operators to output operators</td>
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<tr>
<td>$u$</td>
<td>dimensionless average input field amplitude</td>
</tr>
<tr>
<td>$v_p$</td>
<td>phase-velocity of propagating wave on an infinite transmission line</td>
</tr>
<tr>
<td>$V$</td>
<td>voltage operator along a transmission line</td>
</tr>
<tr>
<td>$w_{mp}^l$</td>
<td>wavelet for wave propagating along direction $l = \rightarrow, \leftarrow$</td>
</tr>
<tr>
<td>$w_{mp}$</td>
<td>Shannon wavelet</td>
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<td>$x_{a,b}$</td>
<td>integrated homodyne current for modes $a, b$ in absence of imperfections</td>
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<td>$X$</td>
<td>quadrature of homodyne detection</td>
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<td>$X_{a,b}$</td>
<td>$X$ measurement on mode arnie, bert</td>
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<td>$[X_a X_b]$</td>
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<td>$Y$</td>
<td>quadrature of homodyne detection</td>
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<td>$[Z_a Z_b]$</td>
<td>two-qubit quantum non-demolition $Z_a Z_b$ measurement on modes arnie, bert</td>
</tr>
<tr>
<td>$Z_c$</td>
<td>characteristic impedance of a transmission line</td>
</tr>
<tr>
<td>$</td>
<td>\alpha\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\beta\rangle$</td>
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<tr>
<td>$\epsilon_p$</td>
<td>pump amplitude</td>
</tr>
<tr>
<td>$\eta$</td>
<td>inefficiency parameter</td>
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<td>$\eta_{1(2)}$</td>
<td>inefficiency parameter before (after) two-qubit measurement</td>
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<td>$\Theta(t)$</td>
<td>Heaviside Theta function</td>
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<td>$\kappa_{a,b,c}$</td>
<td>decay rates of modes $a, b, c$</td>
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<td>$\kappa_{r_1,2}$</td>
<td>decay rates of readout modes $a_{r_1,2}$</td>
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<td>$\kappa_{1\text{ph}}$</td>
<td>single-photon dissipation rate</td>
</tr>
<tr>
<td>$\kappa_{2\text{ph}}$</td>
<td>two-photon dissipation rate</td>
</tr>
</tbody>
</table>
\( \kappa_{ps} \) \hspace{1em} \text{parity-selection dissipation rate}

\( \kappa_{eff} \) \hspace{1em} \text{coupled dissipation rate}

\( \lambda \) \hspace{1em} \text{joint-photon-number-modulo-4 outcome}

\( \xi_{a,b} \) \hspace{1em} \text{homodyne quadratures of arnie,bert}

\( \xi \) \hspace{1em} \text{photon annihilation/creation operators}

\( \Xi \) \hspace{1em} \text{homodyne quadratures}

\( \hat{\xi} \) \hspace{1em} \text{photon annihilation/creation operators in rotating frame}

\( \hat{\xi}_{in,\text{out}} \) \hspace{1em} \text{traveling incoming/outgoing field operator under rotating wave approximation in rotating frame}

\( \Pi_{a,b} \) \hspace{1em} \text{single-qubit measurement operator on arnie, bert}

\( \Pi_{ab} \) \hspace{1em} \text{two-qubit measurement operator on arnie and bert}

\( [\Pi_{ab}] \) \hspace{1em} \text{two-qubit quantum non-demolition measurement operator on arnie and bert}

\( \rho_{ABab} \) \hspace{1em} \text{density matrix of Alice, Bob, arnie, bert}

\( \sigma_{x,y,z} \) \hspace{1em} \text{Pauli x,y,z operator}

\( |\phi^{\pm}\rangle \) \hspace{1em} \text{even Bell-states \( (|g,g\rangle \pm |e,e\rangle)/\sqrt{2} \)}

\( \Phi_i \) \hspace{1em} \text{node-flux at node} \( i \)

\( \Phi_0^i \) \hspace{1em} \text{zero-point fluctuation of the flux}

\( \varphi_0 \) \hspace{1em} \text{reduced flux quantum \( [= h/(2e)] \)}

\( \chi_{aa} \) \hspace{1em} \text{self-Kerr of mode} \( a \)

\( \chi_{ab} \) \hspace{1em} \text{cross-Kerr between modes} \( a, b \)

\( |\psi^{\pm}\rangle \) \hspace{1em} \text{odd Bell-states \( (|g,e\rangle \pm |e,g\rangle)/\sqrt{2} \)}

\( \psi_{\mu}, \psi^{\dagger}_{\mu} \) \hspace{1em} \text{photon annihilation/creation operators of the temporal mode} \( \mu \)

\( |\Psi_{ABab}\rangle \) \hspace{1em} \text{state vector of Alice, Bob, arnie, bert}

\( \omega_{a,b,c} \) \hspace{1em} \text{frequencies of modes} \( a, b, c \)

\( \tilde{\omega}_{a,b,c} \) \hspace{1em} \text{renormalized frequencies of modes} \( a, b, c \)

\( \omega_p \) \hspace{1em} \text{frequency of pump of the Josephson Parametric Multiplier}

\( \Omega \) \hspace{1em} \text{frequency of a resonator mode}
## List of Acronyms

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Chapter 1

Prologue

The development of quantum mechanics in the first quarter of the twentieth century was one of the most revolutionary advancements of natural science since the work of Newton some three hundred years earlier. In one sweeping stroke, this theory resolved several questions that had been puzzling physicists of the time: like the 'ultraviolet catastrophe' in the laws governing radiation and the spiraling of the electron into the nucleus in existing theories of the atom. Since then, quantum mechanics has been applied with spectacular success to explain a wide range of physical phenomena, ranging from elementary particle collisions to properties of the electron fluid in condensed matter systems.

Two other great revolutions of the twentieth century were the development of computer technology and information theory. Since Alan Turing's first description of what entails a programmable computer and its first practical implementation by John von Neumann, computing has made tremendous progress. The subsequent discovery of transistors in 1947 led to spectacular development of computing hardware, which led to the famous Moore's law that stated that the number of transistors in an integrated circuit, and thus, crudely speaking, the computer power, doubled every two years for a constant cost. Shortly after Turing and von Neumann, Claude Shannon formulated a mathematical definition of information and computed the maximum amount of information that could be transmitted over a noisy communication channel. Since then, scientists have engaged in improving the encoding of information in order to achieve the Shannon limit. These two revolutions have rendered information processing devices such as personal computers and smartphones ubiquitous in modern daily life.

These three seemingly different strands of research came together when scientists from these
fields started to ask questions about the limitations of computing and communication in the context of quantum machines. First, Richard Feynman, in 1982, posed the following question: “Can physics be simulated by a universal computer?” [1]. He realized that simulating physical systems which are governed by quantum mechanics is inherently inefficient with a classical computer and a computing machine that obeys the laws of quantum mechanics can accomplish the task much more easily. Second, David Deutsch, in 1985, asked if it was possible to provide a foundation for the concept of a universal computing machine using the laws of physics and came to the conclusion that a “...class of model computing machines that is the quantum generalization of the class of Turing machines ...” was indeed feasible [2]. Third, Charles Bennett and Stephen Wiesner, in 1992, addressed the question of efficient transmission of classical information using a quantum channel and discovered superdense coding which comprises of transmitting two classical bits of information by sending only one quantum bit [3] (see below for the definition of a quantum bit).

The aforementioned initial efforts culminated in the discovery by Peter Shor in 1994 that the problem of finding prime factors of large integers, whose complexity is central to modern day cryptosystems, can be efficiently solved by a quantum computer [4]. This was shortly followed by the Grover search algorithm that demonstrated that a quantum mechanical computer could search through an unsorted database much faster than its classical counterpart [5]. At this time, protocols for secure distribution of keys for communication using quantum mechanics had already been discovered by Bennett and Brassard [6] and Ekert [7]. These quantum key distribution protocols demonstrated that harnessing the power of quantum mechanics enabled achieving absolute limits in privacy in communication allowed by the existing laws of physics. All these results emphatically established the power of quantum information processing (QIP) over its classical counterpart, not only as a resource for information processing with groundbreaking implications, but also for its potential to provide new fundamental insights in physics, computer science and information theory.

Since QIP is about how Nature and thus, every physical object, processes information, it is reasonable to ask: what is the best way to realize it experimentally? To answer this question, first we introduce the basic building block of quantum information: a ‘qubit’, which is a contracted form of ‘quantum bit’. It generalizes its classical ‘bit’ counterpart. Unlike a ‘bit’ that has only two possible states 0 and 1, a qubit is characterized by a state that can be any superposition of the two. Conventionally, the states of a qubit are denoted by |0⟩ and |1⟩, and thus, a generic state of a qubit can be written as α|0⟩ + β|1⟩, where α, β are complex numbers with the restriction |α|^2 + |β|^2 = 1 and only the product αβ* and the difference |α|^2 − |β|^2 are meaningful. Experimental implementation of QIP comprises of robust realization of these qubits in physical systems and performing gates and
measurements, in a controlled manner, on a multitude of them. The key challenge in achieving this arises due to the fragility of these qubits. While the qubits need to be interacting with the outside world for gates and measurements to be done on them, at the same time, unwarranted interactions with the environment cause them to lose their quantum properties, a phenomenon known as decoherence. Furthermore, implementing QIP devices in an actual physical systems will inevitably be imperfect and this will give rise to errors in these devices. Akin to correction of the residual errors in their classical counterparts, these QIP devices need a mechanism to correct errors that may occur and limit its functionality [8–10]. This has stimulated a search for the ideal platform for QIP: a carefully designed, controllable physical system which fulfills the aforementioned requirements.

To that end, several proposals have been put forth since the early developments of quantum computing. The necessary requirements for a physical platform to be a viable architecture for QIP were first formulated by DiVincenzo [11]. These were subsequently extended to include error correction requirements.Crudely speaking, what is required is a scalable, controllable quantum system sufficiently isolated from noise, together with the ability to reliably manipulate (initialize, perform universal set of gates on and measure) its qubits and perform error correction on them. Promising candidates that satisfy these criteria include, but are not restricted to, photons [12], trapped atoms [13], trapped ions [14–16], nuclear magnetic resonance molecules [17,18], semiconductor quantum dots [19] and superconducting circuits [20,21]. This thesis is devoted to QIP with continuous variables in superconducting circuits. In what follows, we provide a brief overview how this thesis work fits in with the overall program of QIP with continuous variables at Yale.

In contrast to other platforms for QIP that involve inherently quantum objects (photons, atoms, ions, and molecules), superconducting circuits rely on the quantum mechanical behavior of macroscopic objects [22–24], that are built according to certain engineering specifications. These specifications determine the quantum behavior of these objects rather than the fundamental physical constants. This may sound strange since it would seem that only the behavior of microscopic objects is governed by quantum mechanics while macroscopic objects are described by classical mechanics. The conflict is resolved as follows. Consider a superconducting circuit-element: the LC oscillator, at temperature much below the superconducting transition temperature, when all electrons are condensed to form a defect-free liquid of Cooper pairs. The capacitor and the inductor are fabricated with appropriate parameters so that they are well-approximated by the lumped element description. This results in the LC oscillator being described by a single degree of freedom (the flux through the inductor coil and its conjugate charge on the capacitor plates). The operation temperature should be
lower than that associated with the characteristic frequency of the element so that thermal noise does not destroy the quantum behavior of this oscillator. Further, the width of the different energy levels due to residual dissipation should be much smaller than the level spacings so that individual energy levels are accessible. All of these three criteria are met for superconducting LC oscillators at dilution refrigeration temperature, resulting in these macroscopic objects behaving quantum mechanically.

However, it is not possible to perform universal QIP with continuous variables with only quantum mechanical harmonic oscillators. They have a quadratic hamiltonian in the mode amplitudes and by themselves cannot be controlled to access every state. A necessary ingredient for QIP with continuous variables are nonlinear operations which give rise to states with negativity in their Wigner functions. Thus, we need Hamiltonians with higher than quadratic terms in mode amplitudes [25–30]. Fortunately, superconducting circuit-QED systems possess a strong, tunable, dispersive nonlinearity, arising out of the robust phenomenon known as the Josephson effect. This effect gives rise to the required higher than quadratic Hamiltonian, making QIP feasible in these systems. The approach taken at Yale is the ‘modular architecture’ for QIP [16,31,32], which is briefly outlined below.

The modular architecture for QIP is akin to multicore processors present in today’s classical computers. It has the advantage of being readily scalable to quantum information processing devices that can be of practical use. It was first proposed for a trapped-ion quantum computer and here, we will restrict ourselves to a modular architecture for a quantum computer adapted for superconducting circuits [33]. In this architecture, the quantum computer is composed of modules, each of which contain a memory qubit and a communication qubit. The memory qubit is used for storage and protection of quantum information and the communication qubit serves as an intermediary for communication of information to the outside of the module. Transfer of information between any two arbitrary modules is performed using flying qubits. They can either be directly transmitted using superconducting transmission lines or through a switchable router using directional amplifiers and frequency converters [34–36]. The flying qubits are subsequently measured using single and two-qubit measurement apparatus, which are essential in this architecture for performing two-qubit gates between any two memory qubits (see below).

The memory qubit is a quantum system, in which a qubit is encoded and protected from decoherence using quantum error correction. Traditional approaches of realizing a quantum memory involve redundantly encoding the logical qubit in a larger Hilbert space using auxiliary qubits, with

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1. A linear optical quantum computing scheme has been proposed which uses Fock states of photons, together with linear scattering elements and photodetectors [12].
Figure 1.1: Schematic for a modular architecture of quantum computing. Each module is composed of a memory qubit and a communication qubit. The memory qubit is encoded in the phase-space of a harmonic oscillator. Here, as an example, is shown the case when the logical qubit is encoded in superpositions of Schrödinger cat states of given photon-number-parity [37]. The harmonic oscillator is coupled dispersively to a transmon qubit, which plays the role of the communication qubit. They interact locally through the dispersive interaction. This interaction, together with measurements and gates on the communication qubit, enables single-qubit gates and quantum error correction on the memory qubit. Each communication qubit interacts with a flying qubit. The flying qubit is encoded in the phase-space of a propagating microwave mode. These flying qubits are routed through a switchable router, which is composed of directional amplifiers and frequency converters. The router can route any two flying qubits to the single and two qubit measurement apparatus. By performing single and two qubit measurements on the flying qubits, one remotely entangles any two communication qubits. This enables performance of nonlocal two-qubit gates and state teleportation which are necessary for quantum computation.
different error mechanisms giving rise to different syndromes [8,9,38]. For superconducting circuits, these qubits could be transmon qubits [39,40]. However, there are two main drawbacks of these approaches: i) with each added ancillary qubit, several new decoherence channels are added, ii) experimental implementation of an error correcting scheme involving a large number of qubits remains challenging. In contrast to these approaches, recently, an alternate hardware efficient scheme of error correction has been proposed, in which a qubit is encoded in the infinite dimensional Hilbert space of a harmonic oscillator mode (LC resonator mode) [37,41–43]. With this encoding, the dominant source of error arises out of loss of photons from the resonator. These errors can be tracked and subsequently corrected for, thereby realizing a quantum memory.

The communication qubit is a transmon qubit. In its original incarnation, it is comprised of two superconducting islands coupled through two Josephson junctions [39]. However, here, we consider a three dimensional version introduced in [44] for its superior coherence properties. This transmon qubit interacts dispersively with the harmonic oscillator mode in which the memory qubit is encoded. Measurements and unitary operations on the communication qubit serve both as a way to perform single qubit gates on the memory qubit and to perform quantum error correction on the memory qubit [42,43].

The flying qubit is encoded in the phase-space of a propagating microwave mode. Each flying qubit interacts with its associated communication qubit. Subsequent single and two-qubit measurements on the flying qubits, due to measurement back-action, enable remote entangling of any two communication qubits. This enables two-qubit gates between any two arbitrary memory qubits [31], a necessary requirement for an architecture for a quantum computer [9]. This is where the work in this thesis comes in, which is to investigate concurrent remote entanglement protocols to remotely entangle any two arbitrary communication qubits. In particular, this thesis answers the following questions. *Are there robust, deterministic \(^2\), concurrent remote entanglement protocols using continuous variables of propagating modes of light? What would then be their key characteristics?*

The thesis is organized as follows. In Chapter 2, we introduce the concept of concurrent remote entanglement and provide a summary of the key results obtained in this thesis. These comprise of two variations of a new protocol that was proposed to achieve concurrent remote entanglement, one using the Josephson Parametric Multiplier (JPM) and the other using joint-photon-number measurements. In Chapter 3, we describe the protocol using the JPM, followed by a detailed analysis of the JPM in Chapter 4. This is followed by the description of the protocol using joint-photon-

\(^2\) In this thesis, a (non-)deterministic protocol for concurrent remote entanglement refers to a protocol that generates any maximally entangled Bell-state with (less-than-)unit probability in a single run.
number measurements in Chapter 5. A concluding summary and outlook is provided in Chapter 6.

In addition to these results, several others were obtained during the course of this thesis work. While being relevant to the subject of QIP in superconducting circuits, they are not pertinent to answering the question asked in this dissertation. These are provided in the appendices. In Appendix A, we describe a protocol for continuous generation and stabilization of mesoscopic field superposition states in superconducting circuit-QED systems. Subsequently, in Appendix B, we define the state of a propagating photon without rotating wave approximation. This is followed by Appendix C, which introduces a new wavelet basis spanning the state-space of a harmonic oscillator. In Appendix D, we describe proposals for experimental implementations of nonlinear mode-mixing relevant for QIP using Josephson junctions.

Finally, to make the thesis concise, appendices are provided at the end of the dissertation that describe calculations that are too voluminous to be described in the main chapters. Appendices for Chapters 3 and 5 are provided in Appendices E and F respectively.
Chapter 2

Introduction

2.1 Motivation

Entanglement, a holistic property of compound quantum systems, is one of the quintessential properties of quantum mechanics. While quantum mechanics was mostly developed in the first quarter of the twentieth century, it was not until 1935, when Einstein et al first identified entanglement as the fundamental aspect in which quantum mechanics departed from a local, classical, statistical theory [45]. Shortly thereafter, Schrödinger formulated the mathematical aspects of entanglement in his seminal work [46] and also coined the term ‘Verschränkung’ which means ‘entanglement’. For an entangled compound system, it is the property that “the best possible knowledge of a whole does not necessarily include the best possible knowledge of all its parts, even though they may be entirely separated and therefore virtually capable of being ‘best possibly known’” [47]. This curious property epitomized the departure from the classical way of thinking and seriously troubled physicists. For most of the 20th century, this led to entanglement be largely restricted to analyze foundational aspects of quantum mechanics, including the violation of the Bell-inequality [48–50] and the Bell-Kochen-Specker theorem [51, 52]. It was only in the 1990-s that entanglement was identified as a practical resource, like energy, that could be utilized to accomplish information processing tasks that were impossible using classical means. In fact, entanglement is a key feature of the so-called “quantum parallelism” that provides speedup for certain algorithms in quantum computation [9, 53, 54] and is the necessary ingredient that allows quantum communication protocols [7, 55] to outperform their classical counterparts.

In this thesis, we will concern ourselves with a very special type of entanglement, namely, con-
current remote entanglement. Remote entanglement protocols generate entanglement between two
distant, spatially separated, non-interacting quantum systems. The attribute concurrent refers to
the property that the entanglement is obtained by performing gates and measurements on two ancilla
quantum signals arriving concurrently from the two quantum systems under consideration. Thus, no
information is ever directly exchanged between the two quantum systems. Experimental realizations
of concurrent remote entanglement protocols test against local, non-contextual, hidden-variable de-
scriptions of quantum mechanics, providing loophole-free tests of Bell’s inequalities [49, 56–58]. At
the same time, concurrent remote entanglement is crucial for large-scale quantum information pro-
cessing. For instance, it is necessary for implementation of quantum cryptography using the Ekert
protocol [7], teleportation of unknown quantum states [55] and efficient quantum communication
over a distributed quantum network [59,60]. Moreover, concurrent remote entanglement is a crucial
requirement of a scalable, module-based architecture of quantum computing [16,21,31,32]. The pri-
mary goal of this dissertation was to answer the following questions. Are there robust, deterministic,
concurrent, remote entanglement protocols using continuous variables of propagating modes of light?

What would then be their key characteristics?

Prior to describing our protocol, we briefly outline existing relevant works that have proposed
heralded, concurrent, remote entanglement schemes. In order to entangle two, distant stationary
qubits, these proposals use two propagating ancilla qubits. Local entanglement is generated between
each stationary qubit and its associated ancilla, followed by a unitary operation on the ancilla qubits
at the signal processing stage. Lastly, single-qubit measurements are made on the ancilla qubits.

These are described below.

First, a long-standing, deterministic, heralded, protocol using a two-qubit unitary operation, in
particular, a CNOT gate, at the signal processing stage, on the flying qubits is well-known [9, 55].
The stationary qubits, Alice and Bob, are initialized to their respective $|+\rangle$ states, while the flying
qubits, arnie and bert, are initialized to their ground states $|g\rangle$. Subsequently, an entangled state
$(|g,g\rangle + |e,e\rangle)/\sqrt{2}$ of Alice (Bob) and arnie (bert) is created by applying a CNOT gate between the
two. The next step in this protocol comprises of applying a CNOT gate on the flying qubits. After
this step, the state of the four qubits is given by:

$$|\Psi_{ABab}\rangle = \frac{1}{2\sqrt{2}} \{(|g,g\rangle + |e,e\rangle)|+\rangle + (|g,g\rangle - |e,e\rangle)|-\rangle\}|g\rangle$$
$$+ \frac{1}{2\sqrt{2}} \{(|g,e\rangle + |e,g\rangle)|+\rangle + (|g,e\rangle - |e,g\rangle)|-\rangle\}|e\rangle,$$  \hspace{1cm} (2.1)

where, the first, second, third and fourth positions in the kets describe the states of Alice, Bob, arnie
Figure 2.1: Schematic of the remote entanglement protocol using CNOT gate on flying qubits. In the first step, two stationary qubits, Alice and Bob, are initialized to their respective $|\pm\rangle$ states, while two propagating ancilla qubits, arnie and bert, are initialized to their respective $|g\rangle$ states. In the second step of the protocol, local entanglement is generated between Alice (Bob) and arnie (bert), giving rise to the state $(|g,g\rangle + |e,e\rangle)/\sqrt{2}$. This is achieved by a CNOT gate between Alice (Bob) and arnie (bert). In the third step, a two-qubit unitary operation, a CNOT gate, is performed on arnie and bert. This gives rise to the following state for the four-qubits:

$$|\Psi_{ABab}\rangle = \frac{1}{2\sqrt{2}} \left\{ \left( |g,g\rangle + |e,e\rangle \right) |+\rangle |g\rangle + \left( |g,g\rangle - |e,e\rangle \right) |-\rangle |g\rangle 
+ \left( |g,e\rangle + |e,g\rangle \right) |+\rangle |e\rangle + \left( |g,e\rangle - |e,g\rangle \right) |-\rangle |e\rangle \right\}. \tag{2.1}$$

In the last step, two single-qubit measurements (outcomes) $X_a$ ($p_a = \pm 1$) and $Z_b$ ($p_b = \pm 1$) are performed on arnie and bert respectively. Conditioned on the two measurement outcomes $p_a, p_b$, Alice and Bob are projected onto an entangled state $|\Psi_{p_a,p_b}\rangle$.

and bert. In the final step, single-qubit measurements (outcomes) $X_a$ ($p_a = \pm 1$) and $Z_b$ ($p_b = \pm 1$) are performed on arnie and bert respectively. From Eq. (2.1), it follows that, conditioned on the outcomes $p_a, p_b$, Alice and Bob are projected onto an entangled state. In particular, their state is given by:

$$|\Psi_{p_a,p_b=-1}\rangle = \frac{1}{\sqrt{2}} \left( |g,e\rangle + p_a |e,g\rangle \right), \tag{2.2}$$

$$|\Psi_{p_a,p_b=1}\rangle = \frac{1}{\sqrt{2}} \left( |g,e\rangle + p_a |e,g\rangle \right). \tag{2.3}$$

While this scheme generates remote entanglement with unit probability in absence of imperfections, it suffers from the following drawbacks: i) implementing a CNOT gate on propagating modes of light is experimentally challenging, ii) in presence of imperfections, this scheme will necessarily be susceptible to inefficiencies in the CNOT gate on the flying qubits.
Second, prior to this thesis work, a non-deterministic, linear optical quantum computing scheme for heralded, concurrent, remote entanglement generation using single photon states had been proposed [12, 61–63]. In this scheme, the ancilla qubit states are encoded in the Fock states \( |0\rangle, |1\rangle \) of propagating temporal modes. In the first step, local entanglement is generated between each stationary qubit with its associated ancilla. Subsequently, a unitary linear scattering transformation is performed on these flying qubits at the signal processing stage using a beam-splitter, which erases the ‘which qubit information’ i.e. the information about which flying ancilla is entangled to which stationary qubit. Finally, a non-linear detection is performed at the final stage with photodetectors. Conditioned on the outcome of the photodetectors, an entangled Bell-state of the stationary qubits is created. This scheme has been successfully demonstrated in different quantum optical systems [64–68]. However, this scheme suffer from the following drawbacks: i) implementing the local entanglement generation step deterministically is experimentally challenging, ii) even in absence of imperfections, the success-rate for generating maximally entangled states remains less than unity.

Third, simultaneous with this thesis work, a non-deterministic, continuous variable (CV), heralded, concurrent, remote entanglement protocol using a linear, non-degenerate, parametric amplifier was submitted [69]. In fact, its genesis constituted the motivation for this thesis work. In this scheme, the ancilla qubits are encoded in CV states in phase-spaces of propagating modes of light, in particular, in coherent state superpositions. Following initial state preparation and local entanglement between each stationary qubit with its associated ancilla, a two-mode squeezing unitary operation is performed on the ancilla propagating modes at the signal processing stage. Finally, one of the ancilla modes is detected using heterodyne detection. Depending on the outcome of the heterodyne measurement, this scheme gives rise to either a maximally entangled state or an unentangled state with equal probability. This scheme suffers from the following drawbacks: i) it only gives rise to Bell-states with a maximum success probability of \( 1/2 \) in absence of imperfections, ii) this scheme is not amenable to quantum error correction in presence of imperfections.

In this thesis, we take a fundamentally different approach at the signal processing stage to generate concurrent remote entanglement. The ancilla qubits are encoded in coherent state superpositions in phase-spaces of propagating modes of light. Following local entanglement generation of each stationary qubit with its associated ancilla, at the signal processing stage, we use a nonlinear, two-qubit, quantum non-demolition (QND) measurement of these propagating ancillas. This is done by a nonlinear interaction of the propagating modes, followed by a linear detection. At the final stage, we use linear single-qubit CV measurements. These are done by homodyne detections. Depending on the results of these three measurements, the stationary qubits are projected onto a
particular entangled state. As will be shown in the body of the dissertation, this approach has the following advantages: i) our protocol can be straightforwardly implemented in experimental setups, ii) in absence of imperfections, our protocol generates maximally entangled Bell-states with unit probability, iii) in presence of imperfections, performing a two-qubit QND measurement, instead of a two-qubit gate, has the advantage of being able to post-select on events when desired outcomes were obtained, iv) our approach is amenable to quantum error correction to correct for decoherence arising out of imperfections in realistic quantum systems, v) with this approach, one investigates coherent, strong, non-linear interactions of non-classical states of light, which is of interest for fundamental physics.

Superconducting circuit-QED systems have access to excellent sources of coherent states in the form of microwave signal generators and a strong, dispersive and tunable Josephson nonlinearity. Furthermore, the collection and detection efficiencies of these systems can be significantly better than their optical counterparts. This makes them natural candidates to implement our protocol. In fact, sequential remote entanglement using linear microwave signal processing has already been demonstrated for these systems [70].

We describe two variations of our protocol and their CV implementations. The first (second) variation comprises of the following steps: initialization of the four qubits, generation of local entanglement between each stationary qubit with its associated ancilla, a two-qubit, QND measurement $[XX]$ ($[ZZ]$) on the propagating ancillas and finally, a single-qubit measurement $Z$ ($X$) on each of the ancillas. Here, we use $[ ]$ to distinguish the measurements which are QND from the usual quantum measurements. For CV implementation of these variations, we encode the ancilla qubits in even/odd Schrödinger cat states of propagating microwave modes. The local entanglement is generated by a dispersive interaction of each stationary qubit with its associated propagating mode. The QND two-qubit measurement $[XX]$ is performed by a nonlinear interaction of the propagating modes through the Josephson Parametric Multiplier (JPM), followed by a homodyne detection. The QND two-qubit measurement $[ZZ]$ is performed by a joint-photon-numbermodulo-2 measurement using a transmon qubit. The final single-qubit measurements in both variations are performed by homodyne detection of the propagating modes. In absence of measurement imperfections and photon loss, these schemes deterministically generate maximally entangled Bell-states for the stationary qubits. For the sake of convenience, we will henceforth refer to the first and second variations by their respective two-qubit measurements. Thus, the first (second) variation will be called the $[XX]$ ($[ZZ]$) protocol.

While the CV implementations of the aforementioned protocols generate maximally entangled states with unit probability in absence of imperfections, transmission losses and measurement ineffi-
ciencies present in current experimentally accessible quantum systems severely limit the success-rates of our protocols to generate high-fidelity Bell-states. This is because superpositions of coherent states are extremely susceptible to decoherence in presence of photon losses. In contrast, single-photon based remote entanglement schemes make use of the inherent resilience of Fock states to photon loss, which ensures that when a successful event happens, it leads to a very high fidelity entangled state. This motivates a search for a CV concurrent remote entanglement protocol which is resilient to these imperfections. To that end, we introduce a modification of the [ZZ] protocol that increases the success-rate of generating high-fidelity Bell-states. Finally, we describe a different CV implementation of the modified [ZZ] protocol, where we use quantum error correction to correct for the decoherence due to photon loss. In this implementation, we encode the ancilla qubits in superpositions of Schrödinger cat states of a given photon-number-parity. The two-qubit measurement [ZZ] is performed by a joint-photon-number-modulo-4 measurement, while the single-qubit measurements $X$ are performed by homodyne detections.

The chapter is organized as follows. First, we describe our protocols in terms of propagating ancilla qubits in Sec. 2.2. Subsequently, we describe continuous-variable implementations of our protocols in Sec. 2.3. We discuss the resilience of our protocol to finite quantum efficiency and undesired photon loss in Sec. 2.4. Subsequently, we discuss an improvement of our [ZZ] protocol in Sec. 2.5 to improve the success-rate of our protocol. Then, in Sec. 2.6, we discuss an implementation of the modified [ZZ] protocol where we use quantum error correction to suppress the loss of coherence due to photon loss. We summarize and allude to future directions in Sec. 2.7.

### 2.2 Protocols using propagating ancilla qubits

In this section, we describe the [XX] and [ZZ] protocols to entangle two stationary, mutually non-interacting qubits, Alice (A) and Bob (B), using two propagating ancilla qubits, arnie (a) and bert (b). The first step, initialization (INIT), of both protocols comprises of initializing Alice (arnie) and Bob (bert) in their $|+\rangle$($|-\rangle$) states. The second step, local entanglement generation (LEG), for both protocols, comprises of generating local entanglement between Alice (Bob) and arnie (bert) by applying a CPHASE gate between Alice (Bob) and arnie (bert). After this step, the entangled states of Alice (Bob) and arnie (bert) can be written as: $|(g,-) + |e,+\rangle\rangle/\sqrt{2}$. The third step, two-qubit measurement (TQM), of the [XX] ([ZZ]) protocol comprises of making a two-qubit measurement $X_aX_b(Z_aZ_b)$ on arnie and bert, whose outcome is denoted by $p = \pm 1$. For the [XX] protocol, after this measurement, depending on $p$, the four-qubit state is projected on to either one of the
determine the resultant entangled state of Alice and Bob:

arnie and bert. The first, second, third and fourth places in the kets respectively describe the states of Alice, Bob, and the stationary qubits. For the [XX] protocol, this is achieved by performing Z

testimony on arnie, bert respectively denoted by \( \Pi_a \) and \( \Pi_b \) with outcomes \( p_a, p_b \). Here, \( \Pi_a, \Pi_b = Z_a(X_a), Z_b(X_b) \) for the [XX] ([ZZ]) protocol. Conditioned on the three measurement outcomes \( p_a, p_b \), Alice and Bob are projected onto an entangled state \( |\Psi_{p_a,p_b}^p\rangle \). The states of the four qubits at the end of each step is denoted by \( (|\Psi_{ABab}\rangle) \) and is given in Table 2.1.

The following states: \( |\Psi^p=1\rangle = (|g, g, -, -\rangle + |e, e, +, +\rangle)/\sqrt{2} \) or \( |\Psi^p=-1\rangle = (|g, e, -, +\rangle + |e, g, +, -\rangle)/\sqrt{2} \).

Similarly, for the [ZZ] protocol, after this step, the state of the four-qubits is either \( |\Psi'^{p=1}\rangle = (|+, +, g, g\rangle + |-, -, e, e\rangle)/\sqrt{2} \) or \( |\Psi'^{p=-1}\rangle = (|+, -, g, e\rangle + |-, +, e, g\rangle)/\sqrt{2} \).

The fourth and final step, single-qubit measurements are performed on arnie and bert, denoted by \( \Pi_a, \Pi_b \), with measurement outcomes \( p_a, p_b \). Here, \( \Pi_a, \Pi_b = Z_a(X_a), Z_b(X_b) \) for the [XX] ([ZZ]) protocol. Conditioned on the three measurement outcomes \( p_a, p_b, \) Alice and Bob are projected onto an entangled state \( |\Psi_{p_a,p_b}^p\rangle \). The states of the four qubits at the end of each step is denoted by \( (|\Psi_{ABab}\rangle) \) and is given in Table 2.1.
2.3 Continuous variable implementation of the [XX] and [ZZ] protocols

2.3.1 Encoding ancilla qubits in continuous variables

In this section, we describe CV implementations of the [XX] and [ZZ] protocols. To that end, we encode the ancilla qubits in even/odd Schrödinger cat states of propagating modes of microwave light \cite{37,42,71,72}. This is illustrated in Fig. 2.3. The ground (excited) state of each of the ancilla qubits is mapped to an even (odd) Schrödinger cat state of propagating modes of microwave light.

\[ |C_{\alpha}^{\pm}\rangle = N_{\pm}(|\alpha\rangle \pm |\alpha\rangle), \]  

(2.4)

where \( N_{\pm} = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})} \). Consequently, the \(|\pm\rangle\) states of the encoded qubits are approximately mapped to coherent states \(|\pm \alpha\rangle\).
Figure 2.3: Mapping ancilla qubits to even/odd Schrödinger cat states. The ground (excited) state of each ancilla qubit is mapped to even (odd) Schrödinger cat states $|\alpha^+\rangle$ [see Eq. (2.4)] respectively. Consequently, the states $|\pm\rangle$ are approximately mapped to coherent states $|\pm\alpha\rangle$.

### 2.3.2 Initialization

The first step (INIT) of both protocols comprises of initializing the stationary qubits (for definiteness, we will consider transmon qubits [39,40,44]) to their respective $|+\rangle$ states and the propagating ancilla qubits to their $|-\rangle$ states. For the stationary qubits, Alice and Bob, this is done by first resetting them to their respective ground states and then rotating them by $\pi/2$ about their Y directions. For the propagating qubits, arnie and bert, this step is done by generating coherent states $|\alpha\rangle$, with propagating temporal modes, $e^{\kappa_a t/2} \cos(\omega_a t) \Theta(-t)$ and $e^{\kappa_b t/2} \cos(\omega_b t) \Theta(-t)$, where $\omega_a, \omega_b, \kappa_a, \kappa_b$ are defined below. Here, we choose the coherent state amplitudes for arnie and bert to be equal and real without any loss of generality, although it is not necessary for the success of our protocols.

### 2.3.3 Local entanglement generation

The second step (LEG) of both protocols comprises of generating entangled states of the propagating modes with the stationary qubits. To that end, the propagating temporal modes, arnie and bert, are incident resonantly on two cavities, exciting their fundamental modes A and B, with frequencies (decay rates) $\omega_a(\kappa_A)$ and $\omega_b(\kappa_B)$, with $\kappa_a(\kappa_b) \ll \kappa_A(\kappa_B)$. These modes interact dispersively through cross-Kerr interaction [73, 74] with Alice and Bob. This operation is referred to as the conditional...
phase gate. It imparts a qubit-state-dependent phase-shift on the outgoing microwave modes. The resultant entangled qubit-photon states of Alice (Bob) and arnie (bert) is $(|e, \alpha⟩ + |g, -\alpha⟩)/\sqrt{2}$ [75] with temporal profiles $ie^{\omega_a t/2} \cos(\omega_a t)\Theta(-t)$ and $ie^{\omega_b t/2} \cos(\omega_b t)\Theta(-t)$, respectively (see Appendix. E.2 and Chaps. 3,5 of [76]).

2.3.4 Two-qubit measurement

The third step (TQM) comprises of making a two-qubit QND measurement [XX] ([ZZ]) on the propagating modes for the [XX] ([ZZ]) protocol. To that end, the propagating entangled qubit-photon states are first captured in resonators which have fundamental modes $a, b$ with frequencies (decay rates) $\omega_a(\kappa_a), \omega_b(\kappa_b)$. Due to their particular temporal profiles, these flying modes are perfectly captured at $t = 0$. Subsequently, for the [XX] protocol, the two-qubit measurement is performed by a non-linear dissipation of the captured modes, followed by homodyne detection of the nonlinear dissipation channel. For the [ZZ] protocol, the two-qubit measurement is performed by using an auxiliary transmon qubit to measure the joint-photon-number-modulo-2 measurement of the captured modes.

[XX] protocol

The nonlinear dissipation used in this protocol comprises of removing pairs of photons from the modes $a$ and $b$ at rate $\kappa_{2\text{ph}}$. It is realized using a pumped nonlinear interaction of the modes $a, b$ with a low-Q mode $c$ with frequency (decay rate) $\omega_c(\kappa_c)$, together with the dissipation of the mode $c$. The nonlinear interaction is realized using the Josephson Parametric Multiplier (JPM). The JPM comprises of the three resonators and a nonlinear four wave mixing element, the Josephson Four Wave Mixer (JFWM) [Fig. 2.4 (a), (b)]. The JFWM consists of four nominally identical Josephson junctions, as shown in Fig. 2.4 and has four interacting normal modes, which are negligibly shifted in frequency from the original modes $a, b, c$, in the presence of a stiff, off-resonant pump mode with frequency chosen to be $\omega_p = \omega_c - \omega_a - \omega_b$. Thus, under the rotating wave approximation, the mode-mixing arising out of the Josephson nonlinearity leads to an interaction Hamiltonian of the form $H_{\text{int}}/\hbar = i g e^{-i\omega_p t}abc^\dagger + \text{h.c.}$, where $g$, the effective interaction strength, depends on the pump amplitude (see Chapter 4).

If the cavities are designed and pump strength is chosen such that

$$\kappa_a, \kappa_b \ll g, \kappa_{2\text{ph}} \ll \kappa_c,$$  \hspace{1cm} (2.5)
the JPM will provide unidirectional conversion: photons in modes \(a\) and \(b\) are converted into the \(c\) mode, which leaks out before it can be converted back into the \(a\) and \(b\), providing the desired two-photon dissipation channel, characterized by the decay rate \(\kappa_{2\text{ph}} = 4g^2/\kappa_c\) after adiabatic elimination [77]. This nonlinear dissipation channel is monitored with a homodyne detection scheme, which results in the desired two-qubit measurement. The integrated homodyne current falls in one of the two lobes of a continuous bimodal distribution, centered at \(\pm 2g\alpha^2/\kappa_c\). Depending on which lobe the outcome lands, the two-qubit measurement outcome is determined to be \(\pm 1\). The error rate at this step can be controlled by selecting extremal outcomes, beyond some cut-off in each lobe of the distribution. At the end of the measurement, the two-photon dissipation is turned off by switching off the pump at \(\omega_p\). The resultant states at the end of this can be computed analytically

\[
\rho_{\text{p}}^{\pm 1} = \frac{1}{2} \left( \rho_1^{\pm 1} |ee\rangle\langle ee| + \rho_2^{\pm 1} |gg\rangle\langle gg| + \rho_3^{\pm 1} |ee\rangle\langle gg| + \rho_4^{\pm 1} |gg\rangle\langle ee| \right),
\]

(2.6)

\[
\rho_{\text{p}}^{\mp 1} = \frac{1}{2} \left( \rho_1^{\mp 1} |eg\rangle\langle eg| + \rho_2^{\mp 1} |ge\rangle\langle ge| + \rho_3^{\mp 1} |eg\rangle\langle ge| + \rho_4^{\mp 1} |ge\rangle\langle eg| \right),
\]

(2.7)

where the explicit forms of \(\rho_i^{\pm 1}\), \(i = 1, \ldots, 4\) are given in Appendix. E.

[ZZ] protocol

Following the capture of arnie and bert in resonators exciting modes \(a\) and \(b\), a joint-photon-number-modulo-2 i.e. \((n_a + n_b)\mod 2\) is measured [79]. This is done by coupling an auxiliary transmon qubit to the modes \(a, b\). The Hamiltonian governing this interaction is given by

\[
H_{\text{int}} = -\chi (a^\dagger a + b^\dagger b)|e\rangle\langle e|,
\]

where \(\chi\) is the strength of the dispersive cross-Kerr couplings between the transmon and the two modes, taken to be equal \(^1\). This Hamiltonian, together with suitably chosen pulses to the transmon, followed by \(Z\) measurement of the transmon, yields the joint-photon-number-modulo-2 measurement outcome (see [79] and Appendix. F.1). An even (odd) joint-photon-number-modulo-2 outcome corresponds to a measurement result \(p = +1 (p = -1)\). The resultant density matrix for Alice, Bob, arnie and bert is given by

\[
\rho_{\text{p}}^{\pm 1} = |\Psi_{\text{p}}^{\pm 1}\rangle \langle \Psi_{\text{p}}^{\pm 1}|
\]

where

\[
|\Psi_{\text{p}}^{\pm 1}\rangle = \frac{1}{\sqrt{2}} \left( |+, +, C_\alpha^+, C_\alpha^+ \rangle + |-, -, C_\alpha^-, C_\alpha^- \rangle \right),
\]

(2.8)

\[
|\Psi_{\text{p}}^{-1}\rangle = \frac{1}{\sqrt{2}} \left( |+, -, C_\alpha^+, C_\alpha^- \rangle + |-, +, C_\alpha^-, C_\alpha^+ \rangle \right).
\]

(2.9)

\(^1\) Engineering equal cross-Kerr couplings of the transmon to two cavity modes is experimentally challenging and is not a prerequisite for making \((n_a + n_b)\mod 2\) measurement. As was shown in [79], it is possible to accomplish the same for unequal coupling strengths, using higher excited states of the transmon.
Figure 2.4: (a) Schematic of two-qubit measurement apparatus for the continuous-variable implementation of the [XX] and [ZZ] protocols. For the [XX] protocol, the two-qubit measurement is accomplished by the Josephson Parametric Multiplier (JPM), while for the [ZZ] protocol, it is done by a joint-photon-number-modulo-2 measurement apparatus. The propagating modes, arnie and bert, are resonantly incident on the high-Q resonator modes \( a \) and \( b \), and are perfectly captured due to their specific mode-profile (see Sec. 2.3.3). For the [XX] protocol, the modes (frequencies) \( a(\omega_a), b(\omega_b) \), together with a low-Q mode (frequency) \( c(\omega_c) \) and a pump at frequency \( \omega_p = \omega_c - \omega_a - \omega_b \), participate in a non-linear, three-wave interaction \( H_{int}/\hbar = \text{ige}^{-\omega_p t}abc^\dagger + \text{h.c.} \). This nonlinear mode mixing arises out of the Josephson Four Wave Mixer (JFWM). This nonlinear interaction, together with the dissipation of the \( c \) mode, gives rise to a nonlinear dissipation. The nonlinear dissipation channel is subsequently monitored using homodyne detection, realizing the two-qubit measurement. For the [ZZ] protocol, following the capture of arnie and bert in resonators exciting modes \( a \) and \( b \), a joint-photon-number-modulo-2 i.e. \( (n_a + n_b) \mod 2 \) is measured. This is done by coupling an auxiliary transmon qubit to the modes \( a, b \). The Hamiltonian governing this interaction is given by \( H_{int} = -\chi(a^\dagger a + b^\dagger b)|e\rangle\langle e| \), where \( \chi \) is the dispersive cross-Kerr coupling between the transmon and the two modes, taken to be equal and \( |e\rangle \) is the excited state of the transmon. A set of pulses, together with a Z measurement of the transmon [78, 79], realizes the two-qubit measurement in this case. (b) The JFWM has four nominally identical Josephson junctions connected electrically as shown. It has four mutually orthogonal normal (electrical) modes. These normal modes correspond to the cavity modes \( a, b, c \) and the pump mode, which couple non-linearly through the Josephson nonlinearity. (c) Schematic of a transmon qubit used for the joint-photon-number-modulo-2 measurement.
2.3.5 Single-qubit measurement

The fourth and final step of both protocols comprise of homodyne detection of the ancilla microwave modes. They perform the crucial function of disentangling the ancillas from the stationary qubits. The resulting measurements of both $\Xi \in \{X, Y\}$ quadratures of $a$ and $b$ modes result in the system density matrix evolving to:

$$
\rho_{ABab}^P \rightarrow \frac{\mathcal{M}_\Xi \rho_{ABab}^P \mathcal{M}_\Xi^\dagger}{\text{Tr}[\mathcal{M}_\Xi \rho_{ABab}^P \mathcal{M}_\Xi^\dagger]}, \quad \mathcal{M}_\Xi = |\xi_a, \xi_b\rangle \langle \xi_a, \xi_b|.
$$

(2.10)

The post-measurement density matrix of Alice and Bob, $\rho_{AB}$, is computed by tracing out the modes $a$ and $b$.

For the [XX] protocol, the homodyne detection can be made along either the X or Y quadratures of the ancilla modes and it disentangles the ancillas from the stationary qubits in the following way. Consider the case in which the two-qubit parity measurement outcome $p = 1$. The sign of the X quadrature measurements $(x_a, x_b)$ is correlated with the probability that the qubits are in the $|gg\rangle (x_a, x_b < 0)$ or $|ee\rangle (x_a, x_b > 0)$ states and only in certain regions of the $(x_a, x_b)$ plane along the line $x_a = -x_b$ are the two qubit states strongly entangled (see Fig. 2.5, top panels). Conversely, the Y quadrature measurements will give results centered around $y_a = y_b = 0$, are not correlated with the two qubit states, and do not distinguish between them; the result is that Y measurements always entangle the two qubits with a relative phase which interpolates between the even and odd Bell states (see Fig. 2.5, bottom panels). Similar reasoning holds for the two-qubit measurement outcome $p = -1$.

For the [ZZ] protocol, the homodyne measurements are made along the direction $\arg(\alpha)$ of each of the outgoing microwave modes arnie and bert. Since we have chosen $\alpha \in \mathbb{R}$, the X-quadratures of the microwave modes need to be measured. Consider again the case when the two-qubit measurement outcome $p = 1$. From Eqn. (2.8), it is clear that each of the integrated homodyne signals, denoted by $(x_a, x_b)$ fall in the vicinity of $(\pm \alpha, \pm \alpha)$. If the $(x_a, x_b)$ falls in the vicinity of $(\alpha, \alpha)$ or $(-\alpha, -\alpha)$, the resultant states of Alice and Bob is $(|e,e\rangle + |g,g\rangle)/\sqrt{2}$ and if $(x_a, x_b)$ falls in the vicinity of $(\alpha, -\alpha)$ or $(-\alpha, \alpha)$, the resultant states of Alice and Bob is $(|e,g\rangle + |g,e\rangle)/\sqrt{2}$. Similar set of analysis holds for the $p = -1$ outcome.
2.3.6 Probability of success and overlaps to Bell-states

In this subsection, we plot the probability of outcomes and the overlap to the Bell-states for the two protocols.

[XX] Protocol

Fig. 2.5 shows the probability of success and overlap of the state of Alice and Bob to the Bell-state $|\phi^+\rangle$. We choose the coherent state amplitude to be $\alpha = 1$ and consider the case when the two-qubit measurement outcome $p = 1$ and there is perfect quantum efficiency. As mentioned earlier, following the two-qubit measurement, either the X or the Y quadratures of the propagating modes, arnie and bert, can be measured. The top (bottom) row panels correspond to homodyne detection of the X (Y) quadratures of these modes. Corresponding to the probability of outcomes (top left panel), we see that Alice and Bob are projected on to the Bell-state $|\phi^+\rangle$ along the line $x_a = -x_b$ (top right panel). For outcomes $x_a, x_b > 0(< 0)$, Alice and Bob are projected on to the state $|ee\rangle(|gg\rangle)$. However, for the Y measurements, one gets an entangled state for all outcomes (bottom panels). For instance, majority of the events occur around the origin $y_a = y_b = 0$, corresponding to which the state of Alice and Bob is $|\phi^+\rangle$. However, along the line $y_a = y_b$, the state of Alice and Bob changes continuously, indicated by the alternating bright and dark fringes (bottom right panel). Similar set of results hold for the two-qubit measurement outcome $p = -1$ with the state of Alice and Bob alternating between $|\psi^\pm\rangle = (|e, g\rangle + |g, e\rangle)/\sqrt{2}$ and the features of the figures rotated by $\pi/2$.

[ZZ] Protocol

Fig. 2.6 shows the probability distribution $P^p(x_a, x_b)$ of the outcomes of the integrated homodyne currents $x_a, x_b$, together with the overlaps to Bell-states $|\phi^+\rangle, |\psi^+\rangle$ for the case when the joint-photon-number-modulo-2-measurement outcome $p = 1$. We choose $\alpha = 1$ in absence of transmission losses and measurement inefficiency. The probability distribution contains four Gaussian distributions centered around $x_a = \pm\alpha, x_b = \pm\alpha$. For $(x_a, x_b)$ in the vicinity of $(\alpha, \alpha)$ and $(-\alpha, -\alpha)$, the overlap to the Bell-state $|\phi^+\rangle$ tends to 1, while for $(x_a, x_b)$ in the vicinity of $(\alpha, -\alpha)$ and $(-\alpha, \alpha)$, the overlap to the Bell-state $|\psi^+\rangle$ tend to 1. For outcomes along the lines $x_a = 0$ and $x_b = 0$, Alice and Bob are projected onto an equal superposition of $|\phi^+\rangle$ and $|\psi^+\rangle$ and thus, are not entangled. The plots for the case $p = -1$ are identical with $|\phi^+\rangle, |\psi^+\rangle$ replaced by $|\phi^-\rangle, |\psi^-\rangle$. 

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Figure 2.5: Probability of success and overlap to Bell-state $|\phi^+\rangle = (|ee\rangle + |gg\rangle)/\sqrt{2}$ for the [XX] protocol. The top (bottom) row panels correspond to homodyne detection of the X (Y) quadratures of modes $a, b$. We choose $\alpha = 1.0$ and consider the case when the two-qubit measurement outcome $p = 1$ and there is perfect quantum efficiency and zero spurious photon loss. Top left panel shows the probability of outcomes $P^p(x_a, x_b)$ for X measurements, corresponding to which, we see that the two-qubit state is projected on to $|\phi^+\rangle$ for values around the diagonal $x_b = -x_a$ (top right panel). For events occurring in the quadrant $x_a, x_b > 0(< 0)$, the two-qubit state is projected on to $|ee\rangle(|gg\rangle)$. Bottom left panel shows the probability of outcomes $P^p(y_a, y_b)$ for Y measurements. For events around the line $y_a = -y_b$, the two-qubit state is once again projected on to $|\phi^+\rangle$. However, for Y measurements, the phase of the generated Bell state varies continuously, depending on the particular outcome $(y_a, y_b)$, indicated by the existence of alternating bright and dark fringes in fidelity (bottom right panel). For instance, along the line $y_a = y_b$, the two-qubit state oscillates continuously between $|\phi^+\rangle$ and $|\phi^-\rangle$. A similar computation for $p = -1$ shows similar results for X and Y measurements, with $|ee\rangle \rightarrow |eg\rangle, |gg\rangle \rightarrow |ge\rangle$ in the generated entangled states and the features of this figure rotated by $\pi/2$. 

$P^p=1(x_a, x_b)$
Figure 2.6: Probability distribution $P^p(x_a, x_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix, $\rho_{AB}$, to the Bell-states $|\phi^+\rangle = (|g, g\rangle + |e, e\rangle)/\sqrt{2}$, $|\psi^+\rangle = (|g, e\rangle + |e, g\rangle)/\sqrt{2}$ are shown for the case when the joint-photon-number-modulo-2 measurement yields $p = 1$. We choose $\alpha = 1$ and assume absence of measurement imperfections and photon loss. (Left) Probability distribution showing four Gaussian distributions centered at $x_a = \pm \alpha, x_b = \pm \alpha$. (Center and Right) Corresponding overlaps to Bell-states $|\phi^+\rangle$ tend to 1 for $(x_a, x_b)$ in the vicinity of $(\alpha, \alpha)$ and $(-\alpha, -\alpha)$. Similarly, overlaps to Bell-states $|\psi^+\rangle$ tend to 1 for $(x_a, x_b)$ in the vicinity of $(-\alpha, \alpha)$ and $(\alpha, -\alpha)$. For an outcome on the lines $x_a = 0$ and $x_b = 0$, the resultant state of Alice and Bob is an equal superposition of $|\phi^+\rangle$ and $|\psi^+\rangle$ and is not an entangled state. For $p = -1$, the plots are identical with $|\phi^+\rangle$, $|\psi^+\rangle$ is replaced by $|\phi^-\rangle$, $|\psi^-\rangle$.

2.4 Effect of imperfections on the protocols

The dominant source of imperfection in current circuit-QED systems that affect our protocol is photon loss. These losses occur due to photon attenuation on the transmission lines and other lossy devices like circulators and isolators which are necessary for an actual experimental implementation. For this encoding of ancilla qubit in Schrödinger cat states, loss of a photon is equivalent to a bit-flip error on the ancilla qubit. This is because $a|C^\pm_\alpha\rangle \simeq |C^\mp_\alpha\rangle$, where $a$ is the annihilation operator of the propagating temporal mode. This bit-flip error occurs randomly as the entangled qubit-photon states of Alice (Bob) and arnie (bert) propagate from the stationary qubits to the JPM and from thereon to the homodyne detectors. This results in decoherence of the entangled states and the probability of generating a high fidelity Bell-states diminishes drastically. While both the [XX] and [ZZ] protocols are affected by the presence of imperfections in a similar way, in what follows, we describe the performance of the [ZZ] protocol in presence of imperfections. This is because the [ZZ] protocol is amenable to modifications and quantum error correction that allows us to correct for photon loss (see Chapter 5 for full analysis). For details on resilience of the [XX] protocol in presence of imperfections, see Chapter 3.
2.4.1 Probability of success and overlaps to Bell-states in presence of imperfections

Fig. 2.7 shows the probability distribution of outcomes, \( P(q_a, q_b) \), in presence of inefficiencies and the overlaps to Bell-states \(|\phi^\pm\rangle\) for the case when the joint-photon-number-modulo-2 measurement yields \( p = 1 \). Note that we reserve the letter \( x \) (with appropriate subscripts) for the integrated homodyne currents in absence of imperfections while the letter \( q \) (with appropriate subscripts) denotes integrated homodyne currents in presence of imperfections. We choose \( \alpha = 1 \). Without loss of generality, in the rest of the chapter, we assume equal transmission loss and measurement imperfections for arnie and bert \(^2\). The inefficiency parameter is denoted by \( \eta \), chosen to be 0.8. The probability distribution shows four Gaussian distributions centered at \( q_a = \pm \bar{\alpha}, q_b = \pm \bar{\alpha} \), where \( \bar{\alpha} = \sqrt{\eta} \alpha \) (left panel). The corresponding overlap to the Bell-state \(|\phi^+\rangle\) is substantial for \((q_a, q_b)\) in the vicinity of \((\bar{\alpha}, \bar{\alpha})\) and \((-\bar{\alpha}, -\bar{\alpha})\) (center panel), while the overlap to the Bell-state \(|\psi^+\rangle\) is substantial for \((q_a, q_b)\) in the vicinity of \((\bar{\alpha}, -\bar{\alpha})\) and \((-\bar{\alpha}, \bar{\alpha})\) (right panel). Note that the maximum fidelity for outcomes with non-negligible occurrence probability is \( \sim 0.7 \) instead of 1.0 obtained in the case of perfect efficiency (compare Fig. 2.6). For outcomes along the lines \( q_a = 0 \) and \( q_b = 0 \), one does not generate entangled states for reasons similar to the case of perfect efficiency. Similar set of results hold for \( p = -1 \).

2.4.2 Comparison of the total success rate in presence and absence of imperfections

The total success-rate for generation of entangled states can be computed for different cut-off fidelities by integrating the appropriate region of \((x_a, x_b)\) or \((q_a, q_b)\) space of outcomes. In the perfect (imperfect) case, the majority of the outcomes, occurring around \( \pm \alpha(\pm \bar{\alpha}) \), give rise to entangled states, while the events along the lines \( x_a(q_a) = 0 \) and \( x_b(q_b) = 0 \) do not. Thus, in order to have a high total success-rate of generating entangled states, the number of outcomes along the lines \( x_a(q_a) = 0 \) and \( x_b(q_b) = 0 \) should be minimized. This can be done by increasing the size of \( \alpha \) because the probability of obtaining an outcome along these lines goes down exponentially with \( \alpha^2(\bar{\alpha}^2) \). While in the perfect case \( \alpha \) can be made arbitrarily large giving rise to deterministic generation of entangled states, in the imperfect case, too large a value of \( \alpha \) lowers the success-rate. This is because large values of \( \alpha \) are more susceptible to photon-losses. Thus, there is an optimal choice

\(^2\) The general case of unequal inefficiencies is addressed in Chapter 5.
Figure 2.7: Probability distribution \( \bar{P}^p(q_a, q_b) \) of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix \( \rho_{AB} \) to the Bell-states \( |\phi^+\rangle = (|g,g\rangle + |e,e\rangle)/\sqrt{2}, |\psi^+\rangle = (|g,e\rangle + |e,g\rangle)/\sqrt{2} \) are shown for the case when the joint-photon-number-modulo-2 measurement yields \( p = 1 \). We choose \( \alpha = 1 \). We choose equal transmission loss and measurement imperfections for arnie and bert, and denote this inefficiency parameter by \( \eta \), which is chosen to be \( = 0.8 \). (Left) Probability distribution showing four Gaussian distributions centered at \( q_a = \pm \bar{\alpha}, q_b = \pm \bar{\alpha} \), where \( \bar{\alpha} = \sqrt{\eta} \alpha \). (Center and Right) The corresponding overlap to the Bell-state \( |\phi^+\rangle \) is substantial for \( (q_a, q_b) \) in the vicinity of \( (\bar{\alpha}, \bar{\alpha}) \) and \( (-\bar{\alpha}, -\bar{\alpha}) \), while the overlap to the Bell-state \( |\psi^+\rangle \) is substantial for \( (q_a, q_b) \) in the vicinity of \( (\bar{\alpha}, -\bar{\alpha}) \) and \( (-\bar{\alpha}, \bar{\alpha}) \). We note that the maximum fidelity Bell-state that can be obtained for outcomes with non-negligible occurrence probability is \( \sim 0.7 \) instead of 1.0 obtained for perfect efficiency (compare Fig. 2.6). Similar results hold for the case \( p = -1 \).
Figure 2.8: Total probability $P_{\text{total}}$ of success for different cut-off fidelities and different choices of $\alpha$ are plotted for the case of perfect quantum efficiency (left panel, indicated by $\eta = 1$) and the imperfect case (right panel, where we have chosen $\eta = 0.8$). (Left) For $\alpha \approx 0.5$, the probability of generation of entangled states with overlap $> 0.9$ is around 0.5. Increasing $\alpha$ to $\gg 1$, generates perfect entangled states with near-unit probability. (Right) In presence of imperfections, the probability of generating high fidelity Bell-states goes down for small values of $\alpha$. For instance, for $\alpha \approx 0.5$, we generate entangled states with overlap $> 0.9$ to Bell-states with probability $\approx 0.1$. However, increasing $\alpha$ does not lead to a higher success-rate for generating better entangled states for the chosen inefficiency. This is because larger values of $\alpha$ are more susceptible to photon loss. Depending on the desired cut-off fidelity and the efficiency of the experimental setup, there is an optimal choice of $\alpha$ that leads to the maximal success-rate. For instance, in the case shown, if the desired cut-off fidelity is 0.75, one should choose $\alpha \approx 0.7$.

Fig. 2.8 shows the comparison of the total success-probability for the perfect case ($\eta = 1$) and the imperfect case (for which we have chosen $\eta = 0.8$). As explained above, in absence of imperfections, for $\alpha \ll 1$, most of the outcomes occur around the point $x_a = x_b = 0$, for which the resultant state of Alice and Bob are not entangled. However, as $\alpha \gg 1$, Bell-states of Alice and Bob are generated with unit probability. This is because the probability of obtaining outcomes around $x_a = 0$ or $x_b = 0$ approaches zero for $\alpha \gg 1$. For finite quantum efficiency, low values of $\alpha$ do not lead to entangled Bell-states for Alice and Bob for reasons similar to the case of $\eta = 1.0$. Further, in the imperfect case, increasing the value of $\alpha$ does not monotonically increase the success rate of generating Bell-states to unity. This is because larger values of $\alpha$ are more susceptible to photon loss. Thus, for each value of the inefficiency parameter and cut-off fidelity, there is an optimal choice of $\alpha$ for which success-rate of generating Bell-states is maximized. For $\eta = 0.8$ and cut-off fidelity of 0.75, the optimal choice is $\alpha \sim 0.7$. 

of $\alpha$ for a particular efficiency and cut-off fidelity.
2.5 Modification of the [ZZ] protocol for improved success-rate

As pointed out in the previous section, for experimentally accessible values of efficiency ($\eta \in [0.5, 1]$), the optimal choice of the value of $\alpha$ is $< 1$, typically $\sim 0.5$. For such small values of $\alpha$, there are two factors that limit the success-rate of the generating high-fidelity Bell-states. First, the initial entangled state of Alice (Bob) and arnie (bert) differs substantially from the entangled two-qubit state shown in Sec. 2.2. Instead of a maximally entangled state of Alice (Bob) and arnie (bert), what is actually generated is the state $(|+, 0\rangle - \alpha |-, 1\rangle)/\sqrt{1 + |\alpha|^2}$. Here, $|0\rangle, |1\rangle$ denote the Fock states of the propagating temporal modes for each of arnie and bert. Thus, to leading order, the generated state for Alice (Bob) and arnie (bert), for small value of $\alpha$, is a product state with a small correction.

This leads to lower success rate for generation of entangled states using our protocol. Second, due to the small value of $\alpha$, majority of the events at the final homodyne detection step occur around the point $q_a = q_b = 0$ for which the state of Alice and Bob are not entangled. In this section, we propose a modification of the [ZZ] protocol that remedies the first problem by generating maximally entangled states for Alice (Bob) and arnie (bert) for all values of $\alpha$ after step LEG. The second problem will be addressed with the help of quantum error correction (see Sec. 2.6 and Chapter 5).

To that end, we add a Hadamard gate for each of Alice and Bob at the end of the step LEG. In practice, this is implemented by a $\pi/2$ rotation about the Y axis for each of Alice and Bob. The modified protocol is shown below in Fig. 2.9. The first step (INIT) remains the same as earlier. The second step (LEG) now gives rise to entangled states of Alice (Bob) and arnie (bert) given by: $(g, g) + |e, e\rangle)/\sqrt{2}$. The third step (TQM) makes the [ZZ] measurement on arnie and bert as before, giving rise to either $|\Psi^{p=1}\rangle = (|g, g, g, g\rangle + |e, e, e, e\rangle)/\sqrt{2}$ or $|\Psi^{p=-1}\rangle = (|g, e, g, e\rangle + |e, g, e, g\rangle)/\sqrt{2}$ according as the measurement outcome $p = \pm 1$. The fourth and final step (SQM) performs the individual X measurements of arnie and bert, with outcomes $p_a, p_b$ respectively, each $= \pm 1$. Conditioned on the three measurement outcomes $p, p_a, p_b$, Alice and Bob are projected onto an entangled state $|\Psi^{p}_{p_a=p_b=1}\rangle = (|+, +\rangle + p|-, -\rangle)/\sqrt{2}$ or $|\Psi^{p}_{p_a=p_b=-1}\rangle = (|+, -\rangle + p|-, +\rangle)/\sqrt{2}$.

2.5.1 Comparison of the total success rate in presence and absence of imperfections

In this section, we compare the performance of the CV implementation for the modified [ZZ] protocol in presence and absence of imperfections. Using the encoding described in Eq. (2.4) of Sec. 2.3, step
Figure 2.9: Schematic for the modified [ZZ] remote entanglement protocol. The first step of the protocol comprises of initializing the stationary qubits, Alice (in red) and Bob (in green) in their respective $|+\rangle$ states, and the propagating ancilla qubits, arnie (in dark red) and bert (in dark green) in their respective $|−\rangle$ states. The second step generates local entanglement Alice (Bob) and arnie (bert) by applying a CPHASE gate between Alice (Bob) and arnie (bert), followed by a Hadamard rotation on Alice (Bob). After this step, the entangled state of Alice (Bob) and arnie (bert) is $\frac{|g,g,e,e\rangle + |e,g,g,e\rangle}{\sqrt{2}}$. The third step makes a QND two-qubit measurement, $[Z_a Z_b]$, on arnie and bert. Conditioned on the measurement outcome $p = \pm 1$, a four-qubit entangled state is generated: $|\Psi_{p=1}\rangle = \frac{1}{\sqrt{2}}(|g,g,g,e\rangle + |e,e,e,g\rangle)$ or $|\Psi_{p=-1}\rangle = \frac{1}{\sqrt{2}}(|g,e,g,e\rangle + |e,g,e,g\rangle)$. The fourth and final step consists of making single-qubit measurements $X_a, X_b$ on arnie, bert, with measurement outcomes $p_a, p_b = \pm 1$. Conditioned on the three measurement outcomes $p, p_a, p_b$, Alice and Bob are projected onto an entangled state $|\Psi_{p,p_a,p_b=1}\rangle = \frac{1}{\sqrt{2}}(|+,+\rangle + p|−,−\rangle)$ or $|\Psi_{p,p_a,p_b=-1}\rangle = \frac{1}{\sqrt{2}}(|+,−\rangle + p|−,−\rangle)$.
LEG of the modified [ZZ] protocol yields entangled states of the stationary qubit of Alice (Bob) and
propagating microwave mode arnie (bert): \((|g, C_+\rangle + |e, C_-\rangle)/\sqrt{2}\). This specific entangled state can
be generated by first generating this entangled state inside a qubit-cavity system using the protocol
of [80,81]. As in the [ZZ] protocol, we require the temporal profile of the modes of arnie and bert as
they fly away from Alice and Bob to be
\[ e^{\kappa_{\text{a}} \frac{t}{2}} \cos(\omega_{\text{a}} t) \Theta(-t) \]

and
\[ e^{\kappa_{\text{b}} \frac{t}{2}} \cos(\omega_{\text{b}} t) \Theta(-t) \]

respectively. The specific temporal mode profile can be implemented using a Q-switch [82,83]. Note that in the
limit of \(\alpha \to 0\), the entangled state of Alice (Bob) and arnie (bert) reduces to \((|g, 0\rangle + |e, 1\rangle)/\sqrt{2}\)
and remains maximally entangled. Following the capture of these propagating modes, the joint-
photon-number-modulo-2 measurement is performed, followed by the homodyne measurements. In
Fig. 2.10, the total success-rate for generating Bell-states is shown as a function of different cut-off
fidelities and choices of \(\alpha\) in presence and absence of imperfections. We choose \(\eta = 0.8\) for the
case when there are inefficiencies. For the perfect case, for \(\alpha \ll 1\), the probability of generation of
entangled states with an overlap > 0.9 to a Bell-state is \(\sim 0.5\) while for \(\alpha > 1\) for which we generate
entangled states with unit-probability. On the other hand, in the imperfect case, for the choice of
efficiency parameters \(\eta = 0.8\), we see that small values of \(\alpha (\alpha \ll 1)\) give rise to entangled states
with overlaps to Bell-states > 0.7 with a success-rate of 0.3. However, unlike the perfect case, larger
values of \(\alpha\) do not help getting better success-rate for similar or better entangled states because
of photon loss. We see that for different cut-off fidelities and measurement efficiencies, there is an
optimal choice of \(\alpha\), e.g. in the case shown, for a cut-off fidelity of 0.75, the optimal choice for \(\alpha\) is \(\approx 0.7\). Both cases show a substantially improved success probability for \(\alpha \ll 1\) compared to the
[ZZ] protocol (compare Fig. 2.8).

### 2.5.2 Optimized success rate for the modified [ZZ] protocol

As mentioned in the previous subsection, the maximal success-rate for generating Bell-states can
be found for different cut-off fidelities and inefficiencies by optimizing the parameter \(\alpha\). This opti-
mization is done numerically and the resultant optimized success-rate \((P^*)\) is plotted in logarithmic
scale below (cf. Fig. 2.11). As shown, the modified [ZZ] protocol generates entangled states with a
fidelity of \(\sim 0.6\) with near unit probability. However, the probability of generating higher fidelity
Bell-states diminishes rapidly. For instance, for an efficiency parameter of \(\eta \approx 0.9\), the probability
of generating Bell-states with fidelity of \(\sim 0.95\) is effectively zero (the white rectangle in Fig.
2.11). This motivated a different implementation of the modified [ZZ] protocol, which incorporates
quantum error correction to suppress decoherence due to photon loss. This is described below.
Figure 2.10: Total probability $P_{\text{total}}$ of success for different cut-off fidelities and different choices of $\alpha$ are plotted for the case of perfect quantum efficiency (left panel, indicated by $\eta = 1$) and the imperfect case (right panel, where we have chosen $\eta = 0.8$). (Left) For $\alpha \ll 1$, the probability of generation of entangled states with overlap $> 0.9$ is around 0.5. Increasing $\alpha$ to $\gg 1$, generates perfect entangled states with near-unit probability. (Right) In presence of imperfections, for $\alpha \ll 1$, we generate entangled states with overlap $> 0.7$ to Bell-states with probability in excess of 0.3. However, increasing $\alpha$ does not lead to a higher success-rate for generating better entangled states. This is because larger values of $\alpha$ are more susceptible to photon-losses arising out of the imperfections. Depending on the desired cut-off fidelity and the efficiency of an experimental setup, there is an optimal choice of $\alpha$ that leads to the maximal success-rate. For instance, in the case shown, if the desired cut-off fidelity is 0.75, one should choose $\alpha \simeq 0.7$. In both cases, the success-rate of generating entangled Bell-states is much larger than what is obtained in the [ZZ] protocol for $\alpha \ll 1$ (compare Fig. 2.8).
Figure 2.11: Optimized total success-probability $P^*$ for different cut-off fidelities and different choices of the inefficiency parameter $\eta$ for the modified [ZZ] protocol. The optimization is done for the value of the parameter $\alpha$. For low cut-off fidelities around 0.6 and inefficiencies $\eta \geq 0.6$, the protocol successfully generates Bell-states with near-unit probability. However, as the cut-off fidelity is increased, the success-rate diminishes rapidly. For efficiency parameter of $\eta \simeq 0.9$, we see that the probability of generating entangled states with an overlap of 0.95 to the target Bell-state is effectively zero (the white rectangle in the plot). This leads to an alternative implementation of this protocol, where we incorporate quantum error correction (see Sec. 2.6).
2.6 Remote entanglement with quantum error correction

In this section, we propose a different implementation of our modified [ZZ] protocol. In this implementation, we use a different encoding of the ancilla qubits, where the ground (excited) state is mapped to the state $|C^{0(2)\text{mod} 4}_\alpha\rangle$, hereafter referred to as “mod 4 cat states”, of a propagating temporal mode, defined below:

$$
|C^{0\text{mod} 4}_0\alpha\rangle = \frac{1}{\sqrt{2}} \left( \frac{\cos(|\alpha|^2)}{\cosh(|\alpha|^2)} \right) \left( |C^+_\alpha\rangle + |C^+_i\alpha\rangle \right),
$$

$$
|C^{2\text{mod} 4}_\alpha\rangle = \frac{1}{\sqrt{2}} \left( \frac{\cos(|\alpha|^2)}{\cosh(|\alpha|^2)} \right) \left( |C^+_\alpha\rangle - |C^+_i\alpha\rangle \right).
$$

The state $|C^{0(2)\text{mod} 4}_\alpha\rangle$ has photon-number populations in the Fock states $4n(4n+2), n \in \mathbb{N}$, which is indicated by the notation $0(2)\text{mod} 4$. For this encoding, the two-qubit QND [ZZ] measurement is a joint-photon-number-modulo-4 measurement, while the single-qubit measurements are homodyne detections. In absence of imperfections, the joint-photon-number-modulo-4 outcome can be either 0 or 2. Now consider the case when there are imperfections. Photon loss due to these imperfections takes the photon-number-populations of the propagating temporal mode from the even photon-number-parity manifold to the odd-photon-number-parity manifold. This change in parity changes the outcome of the joint-photon-number-modulo-4 measurement. By detecting this change of the joint-photon-number-modulo-4 measurement outcome, we correct for the decoherence of the entangled qubit-photon states due to loss of a photon in either of the ancillas. Furthermore, additional individual photon-number-modulo-2 measurements of the ancillas, in addition to the joint-photon-number-modulo-4 measurement, suppress the loss of coherence due to loss of a single-photon in both the ancilla modes. For detailed analysis of this implementation, see Chapter 5.

2.6.1 Probability of success and overlap to the Bell-states in absence of imperfections

For this encoding, step LEG of the protocol involves generating entanglement between Alice (Bob) and arnie (bert) giving rise to the following states: $(|g, C^{0\text{mod} 4}_\alpha\rangle + |e, C^{2\text{mod} 4}_\alpha\rangle)/\sqrt{2}$. This set of entangled states can be obtained in an analogous manner using the method described in Sec. 2.5.1. After step LEG, the total state of the system, comprising of Alice, Bob, arnie and bert, can be
written as:

$$|\Psi_{ABab}\rangle = \frac{1}{2} (|g, g, C_0^{\text{mod}4}, C_0^{\text{mod}4}\rangle + |e, e, C_0^{\text{mod}4}, C_0^{\text{mod}4}\rangle + |g, e, C_0^{\text{mod}4}, C_2^{\text{mod}4}\rangle + |e, g, C_2^{\text{mod}4}, C_0^{\text{mod}4}\rangle).$$ \hfill (2.12)

Subsequently, these propagating entangled qubit-photon states are then captured in resonators. Then, in step TQM, a joint-photon-number-modulo-4 measurement is performed on these captured modes. In absence of measurement imperfections and photon losses, the joint-photon-number-modulo-4 has two possible outcomes: $\lambda = 0, 2$ (the two-qubit measurement outcome $p$ of Sec. 2.2 can be written as $p = i^\lambda$), corresponding to which the four-mode state can be written as:

$$|\Psi_{ABab}^{\lambda=0}\rangle = \frac{1}{\sqrt{2}} (|g, g, C_0^{\text{mod}4}, C_0^{\text{mod}4}\rangle + |e, e, C_0^{\text{mod}4}, C_0^{\text{mod}4}\rangle),$$ \hfill (2.13)

$$|\Psi_{ABab}^{\lambda=2}\rangle = \frac{1}{\sqrt{2}} (|g, e, C_0^{\text{mod}4}, C_2^{\text{mod}4}\rangle + |e, g, C_2^{\text{mod}4}, C_0^{\text{mod}4}\rangle).$$ \hfill (2.14)

The final step of the protocol comprises of making homodyne detections of arnie and bert and here we choose the X-quadrature of both these modes (similar results can be obtained for other choices). Consider the case when $\lambda = 0$. From Eq. (2.13), it follows that each homodyne detector will have Gaussian distributions centered around $x_a, x_b = 0, \pm \alpha$. It also follows from Eq. (2.13) that for events $(x_a, x_b)$ in the vicinity of $(\pm \alpha, \pm \alpha)$ and $(0, 0)$, the resulting state of Alice and Bob is $|\phi^+\rangle$, while for outcomes in the vicinity of $(0, \pm \alpha)$ and $(\pm \alpha, 0)$, the resulting state of Alice and Bob is $|\phi^-\rangle$. Similar set of reasoning holds for $\lambda = 2$, when the states $|\psi^\pm\rangle$ are generated. Since the state of Alice and Bob depend only on $(|x_a|, |x_b|)$, the resulting overlap distributions respect a four-fold rotational symmetry in the $(x_a, x_b)$ space (see Fig. 2.12).

Fig. 2.12 shows the probability of success and the overlap to the Bell-states $|\phi^\pm\rangle, |\psi^\pm\rangle$ for this implementation of our protocol. We choose $\alpha = 1$, and plot the results for the two possible joint-photon-number-modulo-4 measurement outcomes: $\lambda = 0, 2$, in absence of imperfections and photon loss. We see that for a majority of outcomes in the $(x_a, x_b)$-space, we get one of the four aforementioned Bell-states. The existence of fringes in the plots due to the measurement of the X-quadratures of both arnie and bert, each of which are in superpositions of $|C_0^+\rangle$ and $|C_0^+\rangle$. The size of the fringes decreases with increasing $\alpha$. The overlaps to the odd (even) Bell-states for the case $\lambda = 0(2)$ are identically equal to zero.

Now, consider the case when there are imperfections. To lowest order in photon loss, either arnie or bert can lose a photon. On losing a photon, the state of arnie or bert goes from $|C_0^{0(2)\text{mod}4}\rangle$ to
Figure 2.12: Probability distribution $P^\lambda(x_a,x_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix $\rho_{AB}$ to the Bell-states $|\phi^\pm\rangle = (|g,g\rangle \pm |e,e\rangle)/\sqrt{2}$, $|\psi^\pm\rangle = (|g,e\rangle \pm |e,g\rangle)/\sqrt{2}$ are shown. We choose $\alpha = 1$ and plot the case of absence of measurement imperfections and photon loss. The top (bottom) left panel shows the probability of outcomes for the joint-photon-number-modulo-4 outcome to be $\lambda = 0(2)$. Corresponding overlaps to the Bell-states $|\phi^\pm\rangle(|\psi^\pm\rangle)$ are plotted in the top (bottom) center and top (bottom) right panels. The overlaps to the odd (even) Bell-states for $\lambda = 0(2)$ are zero and not shown for brevity. We see that for both $\lambda = 0$ and $\lambda = 2$, one gets entangled Bell-states for Alice and Bob for majority of outcomes in the $(x_a,x_b)$ space. The alternating bright and dark fringes in the plots are due to the measurement of X-quadrature of both arnie and bert, both of which are in superpositions of $|C_+^\alpha\rangle$ and $|C_{i\alpha}^+\rangle$. 


the state $|C^{3(1)}_{\alpha}\rangle$ (see Eq. (F.8) and [42] for the definition of $|C^{1,3}_{\alpha}\rangle$). Therefore, when either arnie or bert loses a photon, the joint-photon-number-modulo-4 measurement now yields the values $\lambda = 1$ or $3$, unlike the perfect case outcomes $\lambda = 0$ or $2$ [see Eqs. (2.13), (2.14)]. Thus, measurement of the joint-photon-number-modulo-4 allows us to keep track of loss of a photon in arnie or bert. As will be shown below, this allows us to generate high fidelity entangled states of Alice and Bob with higher success rates than that obtained with the implementation of the modified [ZZ] protocol using joint-photon-number-modulo-2 measurement. In the next order in photon loss, either both arnie and bert lose one photon each or arnie loses two photons, or bert loses two photons. Consider the case in the mod 4 encoding when arnie and bert each lose a photon. Now, the measurement outcome $\lambda$ can be $0$ or $2$ as in the perfect case and just a measurement of $\lambda$ does not reveal if arnie and bert have indeed lost a photon each. However, these events of loss of one photon each in the ancillas can be tracked by individual photon-number-modulo-2 measurements of the ancillas. In this way, we can suppress the loss of coherence due to loss of a photon in both arnie and bert. Note that the other second order or the higher order losses cannot be suppressed by this encoding (see Sec. 2.7).

2.6.2 Optimized success probability for the modified [ZZ] protocol with error correction

The total success-rates for the different values of the quantum efficiency can be computed and are shown in Chapter 5. As shown in Chapter 5, small values of $\alpha$ ($\ll 1$) results in a low success-rates for generating Bell-states. In absence of imperfections, $\alpha$ can be made arbitrarily large to give rise to Bell-states with unit probability. However, in presence of imperfections, larger values of $\alpha$ are more susceptible to photon losses. This limits the success-rate of generating high-fidelity Bell-states. Thus, an optimal value of $\alpha$ present, which is determined by the cut-off fidelity of the generated entangled states and the efficiency parameter. This is shown in Fig. 2.13. Here, we plot the optimized success-rate for the generation of Bell-states for different cut-off fidelities and values of the efficiency parameter for the error-correcting implementation of the modified [ZZ] protocol. We see that not only does this implementation generate entangled states with overlap $\sim 0.75$ with near-unity success-rate for inefficiency of $\eta = 0.9$, but also is able to give rise to Bell-states with overlap $\geq 0.95$ with a success-rate of $\sim 10^{-2}$ (the black rectangle in Fig. 2.13). This clearly demonstrates the advantage of using remote entanglement with error correction in presence of inefficiencies (compare the results of Fig. 2.11 for the protocol without error correction).
Figure 2.13: Optimized total success-probability $P^*$ for different cut-off fidelities and different choices of the inefficiency parameter $\eta$ for the modified [ZZ] protocol with error correction. The optimization is done for the value of the parameter $\alpha$. For low cut-off fidelities around 0.6 and inefficiencies $\eta \geq 0.6$, the protocol successfully generates Bell-states with near-unit probability. Further, this implementation of our modified [ZZ] protocol shows substantially more resilience to finite photon loss compared to the implementation without error correction. For instance, for an efficiency parameter of $\eta \approx 0.9$, we see that the probability of generating entangled states with an overlap of 0.95 to the target Bell-state is about $10^{-2}$ (the black rectangle in the plot). This should be compared to the effectively zero success-rate of generating these states in absence of error correction (compare Fig. 2.11). This protocol does not generate, with nonzero probability, Bell-states with overlaps $\geq 0.9$ for all efficiencies $\geq 0.5$. This is because the mod 4 encoding does not protect against higher than first order photon losses. To that end, encodings proposed in [37, 41, 43, 84] should be used.
2.7 Summary and future directions

To summarize, we have presented a concurrent remote entanglement protocol to entangle two distant, mutually non-interacting, stationary qubits. To that end, we have used a propagating ancilla qubit for each of the stationary qubit. Unlike existing schemes which perform a unitary operation at the signal processing stage to achieve remote entanglement, we have taken a radically different approach to achieve the same. Firstly, we encoded the ancilla qubit states in continuous variable states, in particular in superpositions of coherent states, of propagating modes of light. Secondly, instead of a unitary operation, we used a nonlinear two-qubit QND measurement (\([XX]\) or \([ZZ]\)) on the propagating ancilla qubits to erase the ‘which ancilla qubit is entangled to which stationary qubit information’. Lastly, linear single qubit measurements (X-s or Z-s) were performed on the ancillas. Depending on these three measurement outcomes, the two stationary qubits are projected onto an entangled state.

We used two different continuous variable encodings for the ancilla qubits. First, the ancillas were encoded in even and odd Schrödinger cat states of propagating modes of light. With this encoding, the \([XX]\) measurement was achieved by a nonlinear interaction of the propagating modes, followed by a homodyne detection and the Z measurements were done by homodyne detections. With the same encoding, the \([ZZ]\) measurement was achieved by a joint-photon-number-modulo-2 measurement and the X measurements were performed by homodyne detections. We analyzed the performance of these implementations in presence of finite quantum efficiency and undesired photon loss and described a modification of the \([ZZ]\) protocol to improve success-rate of the generating Bell-states. Second, we described a different implementation of the modified \([ZZ]\) protocol where we used quantum error correction to suppress the loss of coherence due to loss of photons. In this implementation, the ancilla qubits are encoded in mod 4 cat states, the two-qubit measurement is performed by a joint-photon-number-modulo-4 measurement and the single qubit measurements are performed by homodyne detections. The joint-photon-number-modulo-4 measurement enabled correcting photon loss errors to first order. Lastly, we showed that by individual photon-number-modulo-2 measurements of the ancilla qubits, together with the joint-photon-number-modulo-4 measurement, the decoherence due to photon loss can be corrected to a higher order.

We point out to two natural extensions of this work. First, the use of homodyne detection as the single-qubit measurement in the final step of the protocol lowers the success-rate when the encoding for error-correction is used. This is because both arnie and bert are in superpositions of \(|C_{\alpha}^+\rangle\) and \(|C_{i\alpha}^+\rangle\), and thus, irrespective of the choice of the quadrature, the homodyne measurement is always
made on the complementary quadrature of the modes for one of the cat states. This gives rise to fringes in the resultant overlap to the Bell-states and lowers the success rate of generating the same. It will be worthwhile to explore alternatives for the homodyne detection to boost the success rate of the error correcting protocol. Second, the error correcting encoding we used is designed to protect against losses of single-photons to first order. This is why the implementation with error correction protects us against loss of a photon in either of the ancilla qubits. By including individual photon-number-modulo-2 measurements, in addition to the joint-photon-number-modulo-4 measurement, we corrected for the decoherence due to loss of a photon in both of the ancillas. It will be interesting to explore the different encodings of the ancilla qubits for protection against higher order photon losses [37, 41, 43, 84].
Chapter 3

Concurrent remote entanglement using multiplication of quantum signals

3.1 Motivation

Generation of entangled states between spatially separated non-interacting quantum systems is an indispensable ingredient for large-scale quantum information processing [7, 55, 59, 60]. In particular, concurrent remote entanglement, in which propagating quantum signals do not interact with both the systems under consideration, is a desirable feature of a scalable module-based architecture of quantum computing [16, 21, 31, 32]. While existing methods to achieve remote entanglement relied on unitary operations on signals to erase ‘which qubit’ information, our proposed method uses a nonlinear, two-qubit QND measurement of signals coming from each qubit to delete their local orientation. To that end, we employ a ‘multiplication’ of these quantum signals. This multiplication is achieved by a new type of nonlinear signal processing. Josephson junction based superconducting circuit QED systems have access to strong, tunable, purely dispersive nonlinearities, making them natural candidates for implementing this protocol. In fact, sequential remote entanglement with linear microwave signal processing has already been performed using Josephson junction circuits [70].
Figure 3.1: Remote entanglement protocol schematic. The first step of the protocol (INIT) consists of initializing two stationary qubits, Alice (in red) and Bob (in green) to their respective $|+\rangle$ states and the propagating modes, arnie (in dark red) and bert (in dark green) to the state $|-\alpha\rangle$ each. In the second step, LEG, a conditional phase gate, CPHASE, on Alice (Bob) and arnie (bert) is applied, giving rise to entangled states of Alice (Bob) and arnie (bert). In the next step, TQM, the propagating modes, arnie and bert, are captured in resonators exciting ancilla modes $a$ and $b$. Then, a nonlinear interaction of the ancilla modes $a, b$ and parity mode $c$ (implemented by the Josephson Parametric Multiplier (JPM), see below), followed by a homodyne detection (HD$_c$) of the parity mode, effectively realizes a joint two-qubit parity measurement. The resulting qubit-photon state has either even or odd joint qubit parity conditioned on the integrated homodyne current being in either lobe of the distribution (indicated by dashed lines). The last step, SQM, which disentangles the qubits from the photon states, comprises of homodyne measurements of the $a$ and $b$ modes, denoted by HD$_a$ and HD$_b$. Conditioned on the measurement outcome in TQM and SQM, the qubits are projected onto the even or odd Bell manifold, with a relative phase that depends on the measurement outcome in SQM.
3.2 Description of protocol

In the first step (INIT), both Alice and Bob are initialized, using local $\pi/2$ rotations ($Y_{1/2}$), to a superposition of their ground ($|g\rangle$) and excited ($|e\rangle$) states, given by: $(|g\rangle + |e\rangle)/\sqrt{2}$. Propagating modes, with coherent states of amplitude $\alpha$ each, and temporal profile $e^{\kappa_s t/2}\cos(\omega_s t)\Theta(-t)$ and $e^{\kappa_s t/2}\cos(\omega_s t)\Theta(-t)$, are prepared for Marie and Bert. In the next step (LEG), these are incident resonantly on two cavities, exciting their fundamental modes $A$ and $B$, with frequencies (decay rates) $\omega_a(\kappa_A)$ and $\omega_b(\kappa_B)$, with $\kappa_a(b) \ll \kappa_A(B)$. These modes interact dispersively through cross-Kerr interaction [73, 74] with Alice and Bob. This operation is referred to as the conditional phase gate (CPHASE). It imparts a qubit-state-dependent phase-shift on the outgoing microwave modes. The resultant entangled qubit-photon states output from Alice and Bob’s cavities can be written as: $(|e,\alpha\rangle + |g,-\alpha\rangle)/\sqrt{2}$ each [75] with temporal profiles $ie^{\kappa_s t/2}\cos(\omega_s t)\Theta(-t)$ and $ie^{\kappa_s t/2}\cos(\omega_s t)\Theta(-t)$, respectively (see Appendix E.2, [85]). Without loss of generality, we have assumed the coherent state amplitudes to be equal and $\alpha \in \mathbb{R}$, although it is not essential for the success of our protocol.

In the next step (TQM), we realize a joint two-qubit parity measurement by first capturing the propagating modes in resonators and then employing a nonlinear dissipation process. To that end, we introduce the Josephson Parametric Multiplier (JPM) (see Fig. 3.2). The JPM comprises three resonators and a nonlinear four wave mixing element, the Josephson Four Wave Mixer (JFWM). The three resonators have fundamental modes (frequencies, decay rates) $a(\omega_a, \kappa_a), b(\omega_b, \kappa_b)$ and $c(\omega_c, \kappa_c)$. The outputs of cavity modes $A$ and $B$, after propagating through transmission lines, act as inputs to the $a$ and $b$ modes, respectively. Due to their particular temporal profiles, these flying modes are perfectly captured at $t = 0$. A coupled two-mode dissipation is then turned on at $t = 0$, which removes pairs of photons from the $a$ and $b$ modes at a rate $\kappa_{2ph}$. This dissipation, mediated by the jump operator $ab$, is realized by the JFWM, together with the dissipation of the $c$ mode in the following way.

The JFWM consists of four nominally identical Josephson junctions, as shown in Fig. 3.2 (b) and has four interacting normal modes, which are negligibly shifted in frequency from the original modes $a, b, c$, in the presence of a stiff, off-resonant pump mode with frequency chosen to be $\omega_p = \omega_c - \omega_a - \omega_b$. Thus, under the rotating wave approximation, the mode-mixing arising out of the Josephson nonlinearity leads to an interaction Hamiltonian of the form $H_{\text{int}}/\hbar = ig e^{-i\omega_p t}abc^\dagger + \text{h.c.}$, where $g$, the effective interaction strength, depends on the pump amplitude (see Chapter 4). If the
Figure 3.2: (a) Schematic of the JPM of Fig. 3.1. The modes $a(\omega_a)$, $b(\omega_b)$ and $c(\omega_c)$ when pumped at $\omega_p = \omega_c - \omega_a - \omega_b$ participate in a non-linear, three-wave interaction $H_{int}/\hbar = i ge^{-i\omega_p t} abc^\dagger + h.c.$

This nonlinear mode mixing arises out of the Josephson Four Wave Mixer (JFWM). (b) The JFWM has four nominally identical Josephson junctions connected electrically as shown. The coupled system has four mutually orthogonal normal (electrical) modes (see Chapter 4), which couple non-linearly, corresponding to the cavity modes $a$, $b$, $c$ and the pump mode.

cavities are designed and pump strength is chosen such that

$$\kappa_a, \kappa_b \ll g, \kappa_{2ph} \ll \kappa_c,$$

the JPM will provide unidirectional conversion: photons in modes $a$ and $b$ are converted into the $c$ mode, which leaks out before it can be converted back into the $a$ and $b$, providing the desired two-photon dissipation channel, characterized by the decay rate $\kappa_{2ph} = 4g^2/\kappa_c$ after adiabatic elimination [77]. The nonlinear dissipation channel is monitored with a homodyne detection scheme, denoted by HD$_c$, with phase angle arg$(g\alpha^2)$, which measures the value of the integrated homodyne current $x_c$. By selecting outcomes $x_c$, which will fall on either lobe of the distribution, centered at $\pm2g\alpha^2/\kappa_c$, shown in Fig. 3.1, the qubit-photon state is projected on to the even or odd joint qubit-parity subspace. The error rate at this step can be controlled by selecting extremal outcomes, beyond some cut-off in each lobe of the distribution. At the end of the measurement, the two-photon dissipation is turned off by switching off the pump at $\omega_p$.

The homodyne measurement of $c$ in step TQM, is followed in step SQM (Fig. 3.1) by homodyne measurements of the modes $a$ and $b$, denoted by HD$_a$ and HD$_b$. This last pair of measurements is
crucial because, while the two-mode dissipation projects onto the even or odd qubit-parity subspace, the photons left over in the modes \(a\) and \(b\) after step TQM are in a two-mode squeezed state which remains entangled with the state of the qubits. Step SQM disentangles the qubits from these microwave modes, as follows.

Consider the case in which the two-qubit parity measurement projects the system to the even two-qubit parity subspace. The sign of the X quadrature measurements \(x_a, x_b\) is correlated with the probability that the qubits are in the \(|gg\rangle(x_a, x_b < 0)\) or \(|ee\rangle(x_a, x_b > 0)\) states and only in certain regions of the \((x_a, x_b)\) plane along the line \(x_a = -x_b\) are the two qubit states strongly entangled (Fig 3.3, upper panels). Conversely, Y quadrature measurements will give results centered around \(y_a = y_b = 0\), are not correlated with the two qubit states, and do not distinguish between them; the result is that Y measurements always entangle the two qubits (with a relative phase which interpolates between the even and odd Bell states) (Fig. 3.3, lower panels). Similar reasoning holds for the odd two-qubit parity subspace outcomes of step TQM. Hence, while it is possible to have reasonable success rate by making X measurements on modes \(a\) and \(b\), it is always preferable to measure the Y quadrature for optimal success rate.

While it is possible to perform complete stochastic master equation simulations of the protocol we have just outlined (see Appendix E.6), given the assumed separation of time scales [Eqn. (3.1)], we have used, in Fig. 3.3, a simpler approximate, but accurate, model, which can be solved analytically and provides physical insight. Following the capture of the propagating microwave modes in signal resonators, the state of the system comprised of Alice, Bob and modes \(a, b\), at \(t = 0\), is given by \(\rho_{ABab}(t = 0) = |\psi_{ABab}\rangle \langle \psi_{ABab}|\), where \(|\psi_{ABab}\rangle = (|ee, \alpha, \alpha\rangle + |gg, -\alpha, -\alpha\rangle + |eg, \alpha, -\alpha\rangle + |ge, -\alpha, \alpha\rangle)/\sqrt{2}\). Since the two-qubit parity measurement of step (TQM) depends only on unambiguously inferring which side of the distribution the outcome is on, for the purposes of analytic computation, we separately average over the different outcomes for the two lobes of the distribution (see Fig. 3.1). This amounts to replacing the stochastic evolution of the whole system by separate deterministic Lindblad evolutions for the even and odd qubit parity subspaces. During this evolution, the single photon losses of modes \(a, b\) are negligible due to Eqn. (3.1). The system density-matrix in even/odd qubit-parity subspace \(\rho_{ABab}^{p=\pm 1}(t)\) thus evolves according to:

\[
\frac{d\rho_{ABab}^p(t)}{dt} = \kappa_{2ph} D(ab) \rho_{ABab}^p(t),
\]

where \(\rho_{ABab}^p(t = 0) = |\psi_{ABab}^p\rangle \langle \psi_{ABab}^p|\), \(|\psi_{ABab}^{p=1}\rangle = (|ee, \alpha, \alpha\rangle + |gg, -\alpha, -\alpha\rangle)/\sqrt{2}\) and \(D(O)\rho_{ABab}^p = O \rho_{ABab}^p O^\dagger - (O^\dagger O \rho_{ABab}^p + \rho_{ABab}^p O O^\dagger)/2\) is the
Figure 3.3: Probability distribution of outcomes, overlap \( F \) with Bell-state \( |\phi^+\rangle = (|ee\rangle + |gg\rangle)/\sqrt{2} \), concurrence \( C \) of the joint two-qubit system \( \rho_{AB} \) and gradient of overlap are plotted when the qubit-photon state is projected onto one with even two-qubit parity after the second step of the protocol.

The top (bottom) row panels correspond to homodyne detection of the X (Y) quadratures of modes \( a, b \). We choose \( \alpha = 0.75 \) and assume perfect quantum efficiency and zero spurious photon loss. Panel (a) shows the probability of outcomes \( P(x_a, x_b) \) for X measurements, corresponding to which, we see that the two-qubit state is projected on to \(|\phi^+\rangle\) for values around the diagonal \( x_b = -x_a \) (panel (b)). For events occurring in the quadrant \( x_a, x_b > 0(<0) \), the two-qubit state is projected on to \(|ee\rangle\) or \(|gg\rangle\). Panel (c) shows the concurrence \( C(x_a, x_b) \) of \( \rho_{AB} \) for the different outcomes, which varies from 0 (for \( \rho_{AB} \) being a separable state \( |ee\rangle \) or \( |gg\rangle \)) to 1 (in case of maximum entanglement). Panel (d) shows the gradient of the overlap as a function of \( x_a, x_b \), which is zero (in green) for a narrow region around the line \( x_a = -x_b \) where an entangled state is obtained. It changes rapidly on moving away from the line \( x_a = -x_b \) and goes back to zero when the qubit state is projected on to \(|ee\rangle\) or \(|gg\rangle\). Panel (e) shows the probability of outcomes \( P(y_a, y_b) \) for Y measurements. For events around the line \( y_a = -y_b \), the two-qubit state is once again projected on to \(|\phi^+\rangle\). However, for Y measurements, the phase of the generated Bell state varies continuously, depending on the particular outcome \( (y_a, y_b) \), indicated by the existence of alternating bright and dark fringes in fidelity (panel (f)). For instance, along the line \( y_a = y_b \), the two-qubit state oscillates continuously between \(|\phi^+\rangle\) and \(|\phi^-\rangle\). Panel (g) shows the concurrence \( C(y_a, y_b) \) of \( \rho_{AB} \), which is \( \approx 1 \) for all measurement outcomes, indicating generation of maximal entanglement for all outcomes \( (y_a, y_b) \). The small regions of low entanglement are artifacts of the analytic approximation which replaces stochastic evolution of the system with a deterministic Lindblad evolution (see further discussion in the text). Panel (h) shows the gradient of overlap which is zero where the two-qubit state is projected on to \(|\phi^\pm\rangle\) and changes rapidly as the two-qubit state oscillates between \(|\phi^+\rangle\) and \(|\phi^-\rangle\).

The quasi-steady state at the end of this evolution, denoted by \( \rho_{ABab}^{p=\pm 1} \) (see Appendix E.3), subsequently evolves under the single photon loss of \( a \) and \( b \) that are monitored by HD\(_a\) and HD\(_b\).
The resulting measurement of both $\Xi \in \{X, Y\}$ quadratures of $a$ and $b$ modes results in the system density matrix evolving to:

$$\rho_{ABab}^p \rightarrow \frac{\mathcal{M}_{\Xi}^p \rho_{ABab} \mathcal{M}_{\Xi}^\dagger}{\text{Tr}[\mathcal{M}_{\Xi} \rho_{ABab} \mathcal{M}_{\Xi}^\dagger]}, \quad \mathcal{M}_{\Xi} = |\xi_a, \xi_b\rangle \langle \xi_a, \xi_b|.$$ (3.3)

The post-measurement two-qubit density matrix $\rho_{AB}$ is computed by tracing out the modes $a$ and $b$. Conditioned on the outcomes in TQM and SQM, the qubits are projected onto an entangled state in the subspace spanned by $\{|gg\rangle, |ee\rangle\}$ or $\{|eg\rangle, |ge\rangle\}$. The continuous nature of entanglement generation appears as a relative complex amplitude of the two terms of the Bell state, which is determined by the measurement outcome in HD$_a$ and HD$_b$.

In Fig. 3.3, we show the probability of outcomes, overlap with the Bell-state $|\phi^+\rangle = (|ee\rangle + |gg\rangle)/\sqrt{2}$, concurrence and gradient of the overlap for either X measurements (top row panels) or Y measurements (bottom row panels) of the $a$ and $b$ modes. For X measurements, we see that the majority of the events occur for either $x_a, x_b > 0$ or $x_a, x_b < 0$, hence projecting the qubit state onto product states $|ee\rangle$ or $|gg\rangle$. However, the (non-negligible) number of outcomes near the line $x_a = -x_b$, do project the qubit onto the entangle state $|\phi^+\rangle$. Accordingly, the concurrence $C(x_a, x_b)$ varies from 0 (for $\rho_{AB}$ separable) to 1 (in case of maximum entanglement). The width of the region in phase-space where entanglement is generated is a function of $\alpha$ and decreases as $\alpha$ is increased.

The rate of variation of entanglement is indicated by gradient of the overlap $|\phi^+\rangle$ which varies most rapidly perpendicular to the line $x_a = -x_b$. For Y measurements, in contrast, a maximally entangled state is generated for all outcomes, with the phase of the generated Bell state varying continuously in the form $|\phi^\varphi\rangle = (|ee\rangle + e^{i\varphi}|gg\rangle)/\sqrt{2}$, giving concurrence equal to unity at all points. An increase in $\alpha$ makes this variation more rapid. In this case the gradient of overlap is not a measure of entanglement, but just describes the variation of the phase, $\varphi(y_a, y_b)$. A similar computation for the odd manifold shows similar results for X and Y measurements, with $|ee\rangle \rightarrow |eg\rangle, |gg\rangle \rightarrow |ge\rangle$ with the features in the Fig. 3.3 rotated by $\pi/2$.

As mentioned before, our simplified analytic model does not track the precise value of the homodyne current $x_c$, but only its sign. This corresponds to averaging over different outcomes with even/odd qubit parity, leading to Eqn. (3.2) making the quasi-stationary state $\rho_{ABab}^{p=\pm 1}$ slightly impure. This leads to impurity in the post-measurement qubit state for some regions in the outcome plane and is therefore an artifact of the approximation. The small regions of spurious zero concurrence in Fig. 3.3 are due to this. As $\alpha$ is increased, the impurity due to the Lindblad evolution increases. This restricts the accuracy of the approximate analytical theory to $\alpha \sim 1$. For $\alpha \gg 1$, one
Figure 3.4: Maximum overlap with Bell state $|\phi^+\rangle$ is shown upon variation of incident state amplitude $\alpha$ and efficiency parameter $\eta$ for measurement of Y quadratures of a and b. A sample of 500 trajectories were simulated for each data point in the $(\alpha, \eta)$ space and maximum fidelity was noted. For perfect efficiency ($\eta = 1$), it is always possible to generate $|\phi^+\rangle$. The maximum target fidelity goes down as $\eta$ is lowered. Higher values of $\alpha$ is more susceptible to finite efficiency. However, even for $\eta = 0.7$ and $\alpha = 0.5$, we obtain fidelities in excess of 80%, indicating the robustness of the scheme to photon loss and finite quantum efficiency.

can numerically simulate the evolution using the stochastic master equation. A comparison between the analytical solutions and stochastic master equation solutions is provided in Appendix E.6 for $\alpha \sim O(1)$.

3.3 Finite quantum efficiency and photon loss

In what follows, we test the robustness of our protocol to imperfections arising out of undesired photon loss and finite quantum efficiency. We treat these imperfections together as a general efficiency parameter $\eta$. We present here results of numerical simulations for the case when both Y quadratures were measured (Fig. 3.4). Similar results are obtained when X measurements are performed. While for perfect efficiency ($\eta = 1$), it is always possible to generate an entangled Bell state, even for $\eta = 0.7$ and $\alpha = 0.5$, fidelity in excess of 80% is obtained, indicating the robustness of the protocol to these imperfections. However, the photon losses before the JPM prevent a complete trade-off between the success-rate and fidelity as will be discussed in [86]. Note that a low value of $\alpha$ lowers the success rate of entanglement generation since the two-qubit parity measurement in the step TQM relies on unambiguously inferring the location of the homodyne outcome of the parity signal.
3.4 Summary

To summarize, we have presented a protocol for remotely entangling two qubits by performing a set of concurrent quantum operations on propagating microwave modes entangled with the qubits. In contrast to existing schemes based on linear scattering devices, we propose a qualitatively different approach, based on the multiplication of quantum signals to generate remote entanglement. This multiplication is achieved using the Josephson nonlinearity and the high detection efficiency of microwave radiation in circuit QED systems promise a much higher success rate of entanglement generation compared to its optical counterparts.
Chapter 4

Josephson Parametric Multiplier

4.1 Motivation

An essential component of quantum computation with continuous variables (e.g. modes of electromagnetic field) are non-classical states \( i.e. \) states with negativity in their Wigner functions [26–30]. These states can be generated by engineering a Hamiltonian with terms higher than quadratic in mode-amplitude, for instance the Kerr Hamiltonian, which is quartic [87]. The aforementioned Hamiltonian, together with linear scattering elements like beam-splitters, drives and squeezers, is sufficient to perform arbitrary polynomial transformations of the mode variables [25].

Superconducting circuit-QED systems have access to a robust, simple and non-dissipative circuit element: the Josephson tunnel junction, which gives rise to a strong, tunable, non-dissipative non-linear coupling of the electromagnetic modes. The strength of this coupling can be much larger than the linear coupling and dissipation rates of the circuit at the single photon level. Previous works in circuit-QED systems have used special arrangements of Josephson junctions to give rise to parametric amplifiers and frequency convertors [35,36,88–96]. At the heart of these linear scattering devices, are effectively quadratic Hamiltonians that describe the interaction of the driven-dissipative standing microwave modes (see [76,77,97,98]). The operation of amplification or frequency conversion amounts to effectively an linear operations (addition) of quantum signals in the phase-space of these modes.

Here, we go a step further and introduce a new, nonlinear Josephson circuit device that performs four-wave mixing with four distinct modes. As will be shown in this chapter, this device, in presence of a stiff, off-resonant pump, gives rise to a Hamiltonian which is cubic in mode amplitudes
with a tunable coupling strength. Furthermore, under a special choice of frequencies between the modes participating in the nonlinear mode-mixing, this gives rise to a Hamiltonian which effectively performs coherent ‘multiplication’ of two microwave modes. This device is the Josephson Parametric Multiplier (JPM). It consists of three resonators and a nonlinear four-wave mixing device, the Josephson Four Wave Mixer (JFWM). The fundamental modes of these three resonators, together with a stiff, off-resonant pump, participate in a nonlinear mode-mixing. This is described below.

The chapter is organized as follows. First, we describe the JFWM in Sec. 4.2. This is followed by an input-output analysis of the JPM in Sec. 4.3. We derive the Quantum Langevin Equations for this system in Sec. 4.3.1, followed by a semi-classical analysis in Sec. 4.3.2 and computation of the linearized scattering matrix in Sec. 4.3.3. Finally, we summarize our results in Sec. 4.4.

4.2 Josephson Four Wave Mixer

The JFWM consists of four identical Josephson junctions arranged as shown in Fig. 4.1. Operating with a bias current lower than the critical current $I_0$, when the Josephson junctions behave as pure nonlinear inductors, with nonlinear inductance given by: $L_J = \varphi_0/(I_0 \cos \delta)$, where $\delta$ is the gauge-invariant phase of the junction and $\varphi_0 = \hbar/(2e)$ is the reduced flux quantum. We define the node fluxes at nodes $x, y, z, w, u$ as:

$$\Phi_i = \int_{-\infty}^{t} V_i(t) \, dt, \quad i = x, y, z, w, u,$$

where $V_i$ is potential at the node $i$. There are five normal modes of the JFWM, denoted by $\Phi_a, \Phi_b, \Phi_c, \Phi_d, \Phi_e$, given below:

$$\begin{align*}
\Phi_a &= \Phi_x + \Phi_y - \Phi_z - \Phi_w, \\
\Phi_b &= \Phi_x - \Phi_y - \Phi_z + \Phi_w, \\
\Phi_c &= \Phi_x - \Phi_y + \Phi_z - \Phi_w, \\
\Phi_d &= \Phi_x + \Phi_y + \Phi_z + \Phi_w - 4\Phi_u \quad \text{and} \\
\Phi_e &= \Phi_x + \Phi_y + \Phi_z + \Phi_w + \Phi_u.
\end{align*}$$

Out of these five modes, four $\{\Phi_a, \Phi_b, \Phi_c, \Phi_d\}$ participate in a nonlinear mode mixing, while the fifth ($\Phi_e$) remain decoupled from the rest.

The Hamiltonian of the JFWM (denoted by $H_{\text{JFWM}}$) is the sum of the Hamiltonian of each
Figure 4.1: The Josephson Four Wave Mixer (JFWM) has four nominally identical Josephson junctions and has four mutually orthogonal normal (electrical) modes, shown in (a), (b), (c) and (d). Modes (a), (b) and (c) respectively correspond to the ancilla signal a, ancilla signal b and parity signal c modes and (d) corresponds to the stiff, off-resonant pump (see Chapter 3). A fifth mode remains decoupled from the rest and is not shown here for brevity. The Josephson nonlinearity, together with the off-resonant pump, gives rise to the desired three-wave interaction given by $H_{\text{int}}$. 
junction: $-E_J \cos \delta$, where $\delta$ is the gauge-invariant phase of the junction. Hence,

$$H_{\text{JFWM}} = -E_J \left( \cos \frac{\Phi_x - \Phi_a}{\varphi_0} + \cos \frac{\Phi_y - \Phi_u}{\varphi_0} + \cos \frac{\Phi_z - \Phi_u}{\varphi_0} + \cos \frac{\Phi_w - \Phi_u}{\varphi_0} \right)$$  \hspace{1cm} (4.7)$$

where $E_J = \varphi_0 I_0$ is the junction energy. Eq. (4.7) can be rewritten in terms of the normal modes of the JFWM as:

$$H_{\text{JFWM}} = -4E_J \cos \frac{\Phi_a}{4\varphi_0} \cos \frac{\Phi_b}{4\varphi_0} \cos \frac{\Phi_c}{4\varphi_0} \cos \frac{\Phi_d}{4\varphi_0} + 4E_J \sin \frac{\Phi_a}{4\varphi_0} \sin \frac{\Phi_b}{4\varphi_0} \sin \frac{\Phi_c}{4\varphi_0} \sin \frac{\Phi_d}{4\varphi_0}. \hspace{1cm} (4.8)$$

For mode intensities $\Phi_a, \Phi_b, \Phi_c, \Phi_d \ll \varphi_0$, we can ignore terms of order higher than fourth in $H_{\text{JFWM}}$, leading to:

$$H_{\text{JFWM}} = -4E_J \frac{E_j}{8\varphi_0^2} (\Phi_a^2 + \Phi_b^2 + \Phi_c^2 + \Phi_d^2) - \frac{E_j}{1536\varphi_0^4} (\Phi_a^4 + \Phi_b^4 + \Phi_c^4 + \Phi_d^4 + 6\Phi_a^2 \Phi_b^2 + 6\Phi_a^2 \Phi_c^2 + 6\Phi_a^2 \Phi_d^2 + 6\Phi_b^2 \Phi_c^2 + 6\Phi_b^2 \Phi_d^2 + 6\Phi_c^2 \Phi_d^2) + \frac{E_j}{64\varphi_0^4} \Phi_a \Phi_b \Phi_c \Phi_d. \hspace{1cm} (4.9)$$

We see that apart from the desired four wave mixing term: $\Phi_a \Phi_b \Phi_c \Phi_d$, the Hamiltonian has terms quadratic in mode amplitudes, which lead to frequency renormalization and other quartic terms which lead to self and cross Kerr nonlinearities. The three modes $\Phi_a, \Phi_b$ and $\Phi_c$ correspond to the ancilla signal a, ancilla signal b and parity signal modes, while $\Phi_d$ corresponds to a stiff, off-resonant pump (see Chapter 3). The pump enables the four-wave mixing term to be the dominant fourth order interaction term over the Kerr nonlinearities, which will be neglected in subsequent analysis.

In terms of the creation and annihilation operators for these modes, we can write:

$$\Phi_i = \Phi_i^0 (1 + i^i), \hspace{0.5cm} i = a, b, c, d \text{ and } \Phi_i^0 = \sqrt{\langle 0 | \Phi_i^2 | 0 \rangle} \hspace{1cm} (4.10)$$

is the zero point fluctuation of the flux. The JFWM Hamiltonian can be rewritten as:

$$H_{\text{JFWM}} = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c + \hbar \omega_p d^\dagger d + \hbar g_4 (a + a^\dagger)(b + b^\dagger)(c + c^\dagger)(d + d^\dagger), \hspace{1cm} (4.11)$$

where $\omega_a, \omega_b, \omega_c, \omega_p$ are the renormalized frequencies of the modes $a, b, c, d$ and $g_4$ is the effective four-wave interaction strength. The pump frequency is chosen to be:

$$\omega_p = \omega_c - \omega_a - \omega_b, \hspace{1cm} (4.12)$$
which leads under rotating wave approximation (RWA) to:

\[
H_{\text{JFWM}} = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c + \hbar \omega_p d^\dagger d + h g_4 a b c^\dagger d + \text{h.c.} \quad (4.13)
\]

The stiff, off-resonant pump gives rise to an effective three-wave interaction between the modes \(a, b\) and \(c\) with a tunable coupling \(g\), which depends on the pump strength. The leads to an effective Hamiltonian of the JFWM as:

\[
H_{\text{JFWM}} = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c + H_{\text{int}},
\]

\[
H_{\text{int}} = i \hbar g e^{-i \omega_p t} abc^\dagger + \text{h.c.} \quad (4.14)
\]

4.3 Input-Output analysis

In this section, we perform the input-output analysis of the aforementioned Hamiltonian in the case where each of the three resonator modes is coupled to a linear, non-dispersive, semi-infinite transmission line.

4.3.1 Quantum Langevin Equations

Following standard treatments of input-output theory (see [76,85,99], Chap. 3 of [97] and Appendix B of this dissertation), one can write down the Quantum Langevin Equations (QLE-s) for the modes \(a, b\) and \(c\), under the rotating wave, stiff-pump and Markov approximations:

\[
\frac{da}{dt} = -i \omega_a a - \frac{\kappa_a}{2} a - g e^{-i \omega_p t} b^\dagger c + \sqrt{\kappa_a} a^{\text{in}} \quad (4.15)
\]

\[
\frac{db}{dt} = -i \omega_b b - \frac{\kappa_b}{2} b - g e^{-i \omega_p t} a^\dagger c + \sqrt{\kappa_b} b^{\text{in}} \quad (4.16)
\]

\[
\frac{dc}{dt} = -i \omega_c c - \frac{\kappa_c}{2} c - g e^{-i \omega_p t} ab + \sqrt{\kappa_c} c^{\text{in}}, \quad (4.17)
\]

where \(\xi^{\text{in}}, \xi = a, b, c\) denote the incoming traveling photon field operators of the corresponding transmission lines. Without loss of generality, we choose the nonlinear coupling \(g \in \mathbb{R}\). The boundary conditions that traveling and standing photon field operators satisfy are given by:

\[
\sqrt{\kappa_c} \xi(t) = \xi^{\text{in}}(t) + \xi^{\text{out}}(t), \quad \xi = a, b, c, \quad (4.18)
\]
where $\xi_{\text{out}}$, $\xi = a, b, c$ denote the outgoing traveling photon field operators. We transform to the rotating frame for each of the aforementioned operators as follows:

$$\tilde{\xi} = \xi e^{i\omega_\xi t}, \tilde{\xi}_{\text{in}} = \xi_{\text{in}} e^{i\omega_\xi t}, \tilde{\xi}_{\text{out}} = \xi_{\text{out}} e^{i\omega_\xi t}, \xi = a, b, c.$$ (4.19)

This leads to the following equation:

$$\frac{d}{dt}\tilde{a} = -\frac{\kappa_a}{2} \tilde{a} - g\tilde{b}^\dagger \tilde{c} + \sqrt{\kappa_a} \tilde{a}_{\text{in}}$$ (4.20)

$$\frac{d}{dt}\tilde{b} = -\frac{\kappa_b}{2} \tilde{b} - g\tilde{a}^\dagger \tilde{c} + \sqrt{\kappa_b} \tilde{b}_{\text{in}}$$ (4.21)

$$\frac{d}{dt}\tilde{c} = -\frac{\kappa_c}{2} \tilde{c} + g\tilde{a} \tilde{b} + \sqrt{\kappa_c} \tilde{c}_{\text{in}}.$$ (4.22)

Using Eq. (4.18), we can eliminate the standing mode operators in favor of the input and output ones:

$$\left(\frac{d}{dt} + \frac{\kappa_a}{2}\right)\tilde{a}_{\text{out}} = -\left(\frac{d}{dt} - \frac{\kappa_a}{2}\right)\tilde{a}_{\text{in}} - \frac{g\sqrt{\kappa_a}}{\sqrt{\kappa_b}\kappa_c} (\tilde{b}_{\text{in}}^\dagger + \tilde{b}_{\text{out}}^\dagger) (\tilde{c}_{\text{in}} + \tilde{c}_{\text{out}})$$ (4.23)

$$\left(\frac{d}{dt} + \frac{\kappa_b}{2}\right)\tilde{b}_{\text{out}} = -\left(\frac{d}{dt} - \frac{\kappa_b}{2}\right)\tilde{b}_{\text{in}} - \frac{g\sqrt{\kappa_b}}{\sqrt{\kappa_a}\kappa_c} (\tilde{a}_{\text{in}}^\dagger + \tilde{a}_{\text{out}}^\dagger) (\tilde{c}_{\text{in}} + \tilde{c}_{\text{out}})$$ (4.24)

$$\left(\frac{d}{dt} + \frac{\kappa_c}{2}\right)\tilde{c}_{\text{out}} = -\left(\frac{d}{dt} - \frac{\kappa_c}{2}\right)\tilde{c}_{\text{in}} + \frac{g\sqrt{\kappa_c}}{\sqrt{\kappa_a}\kappa_b} (\tilde{a}_{\text{in}} + \tilde{a}_{\text{out}}) (\tilde{b}_{\text{in}} + \tilde{b}_{\text{out}}).$$ (4.25)

Next, we perform a semi-classical analysis of the above set of equations [Eqs. (4.23), (4.24), (4.25)] in steady state. This is followed by a analysis of the quantum fluctuations about these steady-state semi-classical values.

### 4.3.2 Semi-classical analysis

It is simpler to solve Eqs. (4.20), (4.21), (4.22). In steady state, these reduce to:

$$\dot{\tilde{a}} = \frac{2}{\sqrt{\kappa_a}} \tilde{a}_{\text{in}}^\dagger - \frac{2g}{\kappa_a} \tilde{b}^\dagger \tilde{c},$$ (4.26)

$$\dot{\tilde{b}} = \frac{2}{\sqrt{\kappa_b}} \tilde{b}_{\text{in}}^\dagger - \frac{2g}{\kappa_b} \tilde{a}^\dagger \tilde{c},$$ (4.27)

$$\dot{\tilde{c}} = \frac{2}{\sqrt{\kappa_c}} \tilde{c}_{\text{in}}^\dagger + \frac{2g}{\kappa_c} \tilde{a} \tilde{b},$$ (4.28)

where $\tilde{a}, \tilde{b}, \tilde{c}$ refer to the semi-classical values of the operators $a, b, c$ in steady state. In what follows, we describe the case when $\kappa_a = \kappa_b = \kappa_c = \kappa$ and $\tilde{a}_{\text{in}} = \tilde{b}_{\text{in}} = \tilde{c}_{\text{in}} = \sqrt{\kappa} u \in \mathbb{R}, \tilde{c}_{\text{in}} = 0$. We choose this case for its relevance to the coherent multiplication of signals (see Chapter 3). The more general case
can be solved in an analogous manner. For this case, the set of equations reduces to:

\[
\begin{align*}
\tilde{a} \left(1 + \tilde{g}^2 |\tilde{b}|^2\right) &= u, \quad (4.29) \\
\tilde{b} \left(1 + \tilde{g}^2 |\tilde{a}|^2\right) &= u, \quad (4.30) \\
\tilde{c} &= \tilde{g}\tilde{a}\tilde{b}, \quad (4.31)
\end{align*}
\]

where \( \tilde{g} = \frac{2g}{\kappa} \). Eqs. (4.29), (4.30), (4.31) imply that \( \tilde{a}, \tilde{b} \in \mathbb{R} \). Solving this set of equations and performing linear stability analysis, gives us the stable solutions for the variables \( \tilde{a}, \tilde{b} \):

\[
\begin{align*}
\tilde{a} &= \tilde{b}, \quad \tilde{a} + g^2 \tilde{a}^3 = u, \quad \tilde{g} < \frac{2}{u}, \quad (4.32) \\
\tilde{a} &= \frac{u \pm \sqrt{u^2 - \frac{4}{g^2}}}{2}, \quad \tilde{b} = \frac{u \mp \sqrt{u^2 - \frac{4}{g^2}}}{2}, \quad \tilde{g} > \frac{2}{u}. \quad (4.33)
\end{align*}
\]

In Eq. (4.32), only the real root of the cubic equation contributes to the solution. The multi-valued solution is reminiscent of the bi-stable behavior observed in non-degenerate parametric amplifiers (more on this below). The output field amplitudes can be obtained from the above solutions, using the input-output relations. This leads to:

\[
\begin{align*}
\tilde{a}^{\text{out}} &= \sqrt{\kappa}(\tilde{a} - \frac{u}{2}), \quad (4.34) \\
\tilde{b}^{\text{out}} &= \sqrt{\kappa}(\tilde{b} - \frac{u}{2}), \quad (4.35) \\
\tilde{c}^{\text{out}} &= \sqrt{\kappa}\tilde{g}\tilde{a}\tilde{b}. \quad (4.36)
\end{align*}
\]

Fig. 4.2 shows the semi-classical behavior of the output field amplitudes emitted by the resonators for the modes \( \mathbf{a, b, c} \) as a function of \( \tilde{g}u \). The magnitudes of the outputs are normalized by \( \sqrt{\kappa}u/2 \). Consider the case when \( u \neq 0 \). When the pump is switched off \( (\tilde{g} = 0) \), the incident signals at the modes \( \mathbf{a, b} \) are reflected back with unit reflection coefficients, while no photons come out, on average, of the \( \mathbf{c} \) mode. Upon increasing the pump amplitude or the drive strength, we arrive at the point of perfect conversion for \( \tilde{g} = 2/u \), when all the photons of signals exciting the modes \( \mathbf{a, b} \), on average, get converted to the \( \mathbf{c} \) mode, indicated by \( \tilde{a}^{\text{out}} = \tilde{b}^{\text{out}} = 0 \) and \( \tilde{c}^{\text{out}} = \sqrt{\kappa}u/2 \). This is also the point when spontaneous oscillation sets in the system. Unlike a non-degenerate parametric amplifier where the pump mode, whose role is played by \( \mathbf{c} \), is driven with a coherent tone, giving rise to spontaneous oscillations in signal and idler modes (whose roles are played by \( \mathbf{a, b} \)), here, we have the reverse process. The bistable behavior is clear for \( \tilde{g} > 2/u \), when the average output of...
Figure 4.2: Semi-classical output field amplitudes $\tilde{a}_{\text{out}}$ (in red), $\tilde{b}_{\text{out}}$ (in green), $\tilde{c}_{\text{out}}$ (in blue), normalized to $\sqrt{\kappa u}$ are plotted as a function of the effective nonlinearity parameter $\tilde{g}u$. Consider the case when $u \neq 0$. For $\tilde{g} = 0$, which corresponds to the pump being switched off, we see that the output for the $a, b$ modes equal the inputs and no signal comes out of the mode $c$ on average. This is because in absence of the pump, no photons get converted to the $c$ mode. Upon increasing the nonlinear coupling, the point of perfect conversion $\tilde{g} = 2/u$ is reached, when all the photons of signals exciting the modes $a, b$, on average, get converted to the $c$ mode, indicated by $\tilde{a}_{\text{out}} = \tilde{b}_{\text{out}} = 0$. At this point, $\tilde{c}_{\text{out}} = \sqrt{\kappa u}/2$. This is also the point when spontaneous oscillation sets in the system. For $\tilde{g} > 2/u$, the output of the $a$ and the $b$ modes spontaneously take either the red or the green branch. The average output of the $c$ mode equals $\sqrt{\kappa}/\tilde{g}$ [see Eqs. (4.31), (4.33)] and thus, in rescaled units it decreases with increasing $\tilde{g}u$ as an inverse power.

4.3.3 Operator scattering matrix after linearization

The analysis in the previous subsection calculated the average value of the operators. In this section, we perform a linearized analysis of the system in steady state around the obtained semi-classical solutions. Again, we will restrict ourselves to the case considered above: $\kappa_a = \kappa_b = \kappa_c = \kappa$ and
\( \tilde{\xi}^{\text{in}} = \tilde{\eta}^{\text{in}} = \frac{\sqrt{\kappa}}{2} u \in \mathbb{R}, \tilde{c}^{\text{in}} = 0 \). We define the operators around the semi-classical values as follows:

\[
\begin{align*}
\tilde{\xi} &= \tilde{\xi} + \delta \tilde{\xi}, \tilde{\xi}^{\text{in/out}} = \tilde{\xi}^{\text{in/out}} + \delta \tilde{\xi}^{\text{in/out}}, \quad \xi = a, b, c, \\
\sqrt{\kappa} \delta \tilde{\xi} &= \delta \tilde{\xi}^{\text{in}} + \delta \tilde{\xi}^{\text{out}}, \quad \xi = a, b, c.
\end{align*}
\] 

(4.37) (4.38)

From Eqs. (4.20), (4.21), (4.22), we arrive at:

\[
\begin{pmatrix}
\delta \tilde{a} \\
\delta \tilde{a}^\dagger \\
\delta \tilde{b} \\
\delta \tilde{b}^\dagger \\
\delta \tilde{c} \\
\delta \tilde{c}^\dagger
\end{pmatrix}
= \frac{2}{\sqrt{\kappa}}
\begin{pmatrix}
\delta \tilde{a}^{\text{in}} \\
\delta \tilde{a}^{\text{in}\dagger} \\
\delta \tilde{b}^{\text{in}} \\
\delta \tilde{b}^{\text{in}\dagger} \\
\delta \tilde{c}^{\text{in}} \\
\delta \tilde{c}^{\text{in}\dagger}
\end{pmatrix},
\]

(4.39)

where

\[
M =
\begin{pmatrix}
1 & 0 & 0 & \tilde{g} \tilde{c} & \tilde{g} \tilde{b} & 0 \\
0 & 1 & \tilde{g} \tilde{c} & 0 & 0 & \tilde{g} \tilde{b} \\
0 & \tilde{g} \tilde{c} & 1 & 0 & \tilde{g} \tilde{a} & 0 \\
\tilde{g} \tilde{c} & 0 & 0 & 1 & 0 & \tilde{g} \tilde{a} \\
-\tilde{g} \tilde{b} & 0 & -\tilde{g} \tilde{a} & 0 & 1 & 0 \\
0 & -\tilde{g} \tilde{b} & 0 & -\tilde{g} \tilde{a} & 0 & 1
\end{pmatrix}
\]

(4.40)

and \( \tilde{a}, \tilde{b}, \tilde{c} \) are the semi-classical values of the operators \( a, b, c \) obtained earlier. One can then write the output fields as:

\[
\begin{pmatrix}
\delta \tilde{a}^{\text{out}} \\
\delta \tilde{a}^{\text{out}\dagger} \\
\delta \tilde{b}^{\text{out}} \\
\delta \tilde{b}^{\text{out}\dagger} \\
\delta \tilde{c}^{\text{out}} \\
\delta \tilde{c}^{\text{out}\dagger}
\end{pmatrix}
= \begin{pmatrix}
\delta \tilde{a}^{\text{in}} \\
\delta \tilde{a}^{\text{in}\dagger} \\
\delta \tilde{b}^{\text{in}} \\
\delta \tilde{b}^{\text{in}\dagger} \\
\delta \tilde{c}^{\text{in}} \\
\delta \tilde{c}^{\text{in}\dagger}
\end{pmatrix}
\]

(4.41)

\[
S = 2M^{-1} - 1,
\]

(4.42)

where the scattering matrix \( S \) is determined by the normalized pump strength \( \tilde{\eta} \) and the semi-classical values \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \). While \( S \) can evaluated at the parameter regime of interest, its explicit
form of yields no special insight, except at the point of perfect conversion: $\tilde{g} = 2/u$. At this point, $\tilde{a} = \tilde{b} = \tilde{c} = u/2$, for which

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

(4.43)

It can be checked that this matrix is singular, implying the scattering matrix diverges at this point of instability. This indicates that the small-fluctuation approximation of this subsection breaks down at this point, when the spontaneous oscillations set in. This phenomenon is similar to what happens in a parametric amplifier when the gain approaches infinity.

### 4.4 Summary

To summarize, we have presented here a new, four-wave mixing device: the Josephson Parametric Multiplier (JPM). It comprises of three resonators, whose fundamental modes participate in a four-wave mixing interaction in presence of a stiff, off-resonant pump. This nonlinear mode-mixing arises out of the Josephson Four Wave Mixer, which comprises of an arrangement of four-Josephson junctions arranged as in Fig. 4.1. We have described Hamiltonian that governs the JPM and performed semi-classical analysis. Further, we have computed the scattering matrix connecting the input and output fields for small fluctuations around the semi-classical analysis. It is an open problem to find the full scattering relations that connects the input fields to the output fields for this kind of nonlinear scattering.
Chapter 5

Concurrent remote entanglement with quantum error correction

5.1 Motivation

The inevitable presence of imperfections in current experimentally accessible quantum systems have stimulated a search for remote entanglement protocols that are resilient to these imperfections. Heralded remote entanglement schemes based on interference of single photons from distant excited atoms or atomic ensembles using beam-splitters and subsequent photon detection have been proposed [61–63] and demonstrated [64–67]. These protocols make use of the inherent resilience of Fock state to photon loss arising out of imperfections. As a consequence, when a successful event happens, it leads to a very high fidelity entangled state. However, the collection and detection efficiencies limit the success probability of generating entangled states. Alternate protocols using continuous variables of microwave light, in particular superpositions of coherent states, have been proposed that have a high success rate [69,101]. However, in presence of imperfections, the success-rates of these protocols diminishes drastically for generating high fidelity entangled state. This is because superpositions of coherent states are extremely susceptible to decoherence in presence of photon losses. The goal of this chapter is to propose a new, concurrent, continuous-variable, remote entanglement protocol, which is amenable to quantum error correction to suppress the decoherence arising out of photon losses.

The protocol can be summarized as follows. In order to generate entanglement between two distant, stationary qubits, we use a propagating ancilla qubit for each of the stationary qubits.
In the first step, each of the stationary qubits is entangled with its associated propagating ancilla qubit. In the next step, a QND two-qubit measurement $[ZZ]$ is performed on the propagating ancillas. This non-linear measurement erases the ‘which stationary qubit is entangled to which flying qubit information’ and gives rise to four-qubit entangled states. The final step comprises of a single qubit measurement on each of the two ancilla qubits, to disentangle them from the stationary qubits, and finally prepare the desired entangled states between the two stationary qubits.

We describe two continuous-variable implementations of the aforementioned protocol. In the first implementation, we encode the ancilla qubits in Schrödinger cat states of propagating modes of microwave light [37, 42, 71, 72]. The logical basis states of each of the ancilla qubits are mapped to even and odd Schrödinger cat states, denoted by $|C_\alpha^\pm\rangle$, defined below:

$$
|C_\alpha^\pm\rangle = \mathcal{N}_\pm \left(|\alpha\rangle \pm |\alpha\rangle\right),
$$

where $\mathcal{N}_\pm = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$. The two-qubit measurement $[ZZ]$ on the ancillas is a joint-photon-number-modulo-2 measurement, while the single-qubit measurements are homodyne detections. In absence of imperfections, this protocol gives rise to Bell-states with unit probability. However, in presence of imperfections, photon losses lead to decoherence of the propagating (ancilla qubit) microwave modes which are entangled with the stationary qubits. As a consequence, the success rate of generating high fidelity entangled states is drastically decreased.

To remedy this effect, we propose a second implementation of our protocol. In this implementation, we use a different encoding of the ancilla qubits, where the logical basis states are mapped to the states $|C_\alpha^{0,2\text{mod}4}\rangle$, hereafter referred to as “mod 4 cat states” [37, 42], of a propagating temporal mode, defined below:

$$
|C_\alpha^{0\text{mod}4}\rangle = \frac{1}{\sqrt{2\{1 + \frac{\cos(|\alpha|^2)}{\cosh(|\alpha|^2)}\}}} \left(|C_\alpha^+\rangle + |C_{i\alpha}^+\rangle\right),
$$

$$
|C_\alpha^{2\text{mod}4}\rangle = \frac{1}{\sqrt{2\{1 - \frac{\cos(|\alpha|^2)}{\cosh(|\alpha|^2)}\}}} \left(|C_\alpha^+\rangle - |C_{i\alpha}^+\rangle\right).
$$

The state $|C_\alpha^{0(2)\text{mod}4}\rangle$ has photon-number populations in the Fock states $4n(4n + 2), n \in \mathbb{N}$, which is indicated by the notation $0(2)\text{mod}4$. For this encoding, the two-qubit $[ZZ]$ measurement is a joint-photon-number-modulo-4 measurement, while the single-qubit measurements are homodyne detections. The state-space of the ancilla qubits have populations entirely in the even Fock manifold and thus the joint-photon-number-modulo-4 outcome can be either 0 or 2 in absence of imperfections.
Now consider the case when there are imperfections. Photon loss due to these imperfections takes the photon-number-populations of the propagating temporal mode from the even photon-number-parity manifold to the odd-photon-number-parity manifold. This change in parity changes the outcome of the joint-photon-modulo-4 measurement. By detecting this change of the joint-photon-number-modulo-4 measurement outcome, we correct for the decoherence of the entangled qubit-photon states due to loss of a photon in either of the ancillas. Furthermore, additional individual photon-number-modulo-2 measurements of the ancillas, in addition to the joint-photon-number-modulo-4 measurement, suppress the loss of coherence due to loss of a photon in both the ancilla modes.

This chapter is organized as follows. Sec. 5.2 describes our protocol using propagating ancilla qubits. Sec. 5.3 describes two continuous variable implementations of this protocol for the two different encodings of the ancilla qubits mentioned above. Secs. 5.4.1, 5.4.2 incorporate the effect of imperfections like undesired photon loss and detector inefficiencies for the two implementations. Sec. 5.4.3 compares the resilience of the two implementations to these imperfections. Sec. 5.4.4 discusses the improvement to our protocol by incorporating additional, individual, photon-number-modulo-2 measurements. Our results are summarized and future directions are outlined in Sec. 5.5.

5.2 Protocol using propagating ancilla qubits

In this section, we present a concise description of our protocol to entangle two stationary, mutually non-interacting qubits, Alice (A) and Bob (B), using two propagating ancilla qubits, arnie (a) and bert (b) (cf. Fig. 5.1). First, Alice, Bob are initialized to their |+⟩ states, while arnie, bert are initialized to their |−⟩ states. Local entanglement is generated between Alice (Bob) and arnie (bert), by first applying a CPHASE gate between Alice (Bob) and arnie (bert), followed by a Hadamard gate on Alice (Bob). After this step of the protocol, the state of Alice (Bob) and arnie (bert) is: $(|g, g⟩ + |e, e⟩)/\sqrt{2}$. Subsequently, a QND two-qubit measurement, $[Z_a Z_b]$, is performed on arnie and bert, whose outcome is denoted by $p = ±1$. This measurement gives rise to one of the two four-qubit entangled states: $|Ψ^{p=1}⟩ = (|g, g, g, g⟩ + |e, e, e, e⟩)/\sqrt{2}$ and $|Ψ^{p=-1}⟩ = (|g, e, g, e⟩ + |e, g, e, g⟩)/\sqrt{2}$. Here, the first, second, third and fourth position in the kets belong to the states of Alice, Bob, arnie and bert respectively. The final stage of the protocol comprises of making X-measurements on arnie and bert, the outcomes of which are denoted by $p_a (= ±1)$ and $p_b (= ±1)$ respectively. While the two-qubit measurement successfully erases the 'which qubit information', the ancillas still remain entangled to the stationary qubits. The last step performs the necessary task of disentangling the ancillas from Alice and Bob. Conditioned on the three measurement outcomes $p, p_a, p_b$, Alice and
Figure 5.1: Remote entanglement protocol schematic. The first step of the protocol comprises of local entanglement generation between two stationary, mutually non-interacting qubits, Alice (in red) and Bob (in green), with propagating ancilla qubits, arnie (in dark red) and bert (in dark green). First, Alice, Bob are initialized to their |+⟩ states, while arnie, bert are initialized to their |−⟩ states. Subsequently, a CPHASE gate is applied between Alice (Bob) and arnie (bert), followed by a Hadamard rotation on Alice (Bob). After this step, the entangled state of Alice (Bob) and arnie (bert) is (|g,g⟩ + |e,e⟩)/√2. Next, a two-qubit QND measurement, [ZaZb], is performed on arnie and bert. Conditioned on the measurement outcome p = ±1, a four-qubit entangled state is generated: |Ψ_{p=1}⟩ = 1/√2 (|g,g,g,g⟩ + |e,e,e,e⟩) or |Ψ_{p=-1}⟩ = 1/√2 (|g,e,g,e⟩ + |e,g,e,g⟩). Subsequently, single-qubit measurements (X) are performed on arnie and bert, denoted by Xa, Xb, with measurement outcomes pa, pb = ±1. Conditioned on the three measurement outcomes p, pa, pb, Alice and Bob are projected onto entangled states: |Ψ_{p,pa=1}⟩ = (|+,+⟩ + p|−,−⟩)/√2 or |Ψ_{p,pa=−1}⟩ = (|+,—⟩ + p|−,+)⟩/√2.

Bob get entangled with each other, with the final state being: |Ψ_{p,pa=1}⟩ = (|+,+⟩ + p|−,−⟩)/√2 or |Ψ_{p,pa=−1}⟩ = (|+,—⟩ + p|−,+)⟩/√2.

In what follows, we describe proposals for the continuous variable implementations that realize the aforementioned protocol.

5.3 Implementation using propagating superpositions of coherent states

In this section, we describe two continuous variable implementations of the protocol. The first implementation uses the mapping of the ground (excited) state of the ancilla qubits to even (odd) Schrödinger cat states |Cα_{+}⟩ (cf. Fig. 5.2). Consequently, the states |±⟩ are approximately mapped to coherent states |±α⟩. The [ZZ] measurement is performed by a joint-photon-number-modulo-2...
Figure 5.2: Cat-qubit mapping schematic. (Left) The ground (excited) state of each of the ancilla qubits is mapped to even (odd) Schrödinger cat states $|C^+\alpha\rangle$ (see Eq. (5.1)). Consequently, the states $|\pm\rangle$ are mapped to coherent states $|\pm\alpha\rangle$. In this mapping, $Z_aZ_b$ on the propagating ancilla qubits corresponds to a joint photon-number-modulo-2 measurement of the propagating microwave modes. (Right) The ground (excited) state of each of the ancilla qubits is mapped to mod 4 cat states $|C^{0(2)\text{mod4}}\alpha\rangle$ (see Eq. (5.2)). Consequently, $|\pm\rangle$ are mapped to even cat states $|C^+\alpha\rangle$. In this mapping, $Z_aZ_b$ on the propagating ancilla qubits corresponds to a joint photon-number-modulo-4 measurement of the propagating microwave modes. For both encodings, the single-qubit measurements (X) can be implemented by homodyne detections.

In the first step of the protocol, local entanglement is generated between a stationary transmon qubit, Alice (Bob), and a propagating microwave mode, arnie (bert), giving rise to the following states:

$$
\frac{|g,C^+\alpha\rangle + |e,C^+\beta\rangle}{\sqrt{2}}, \frac{|g,C^+\alpha\rangle - |e,C^-\beta\rangle}{\sqrt{2}}.
$$

This specific entangled state can be generated by first generating this entangled state inside a qubit-cavity system using the protocol proposed in [80] and experimentally demonstrated in [81]. Without loss of generality, we choose $\alpha, \beta \in \mathbb{R}, \alpha = \beta > 0$. We require the temporal profile of the modes of arnie and bert as they fly away from Alice and Bob to be $e^{\kappa_a t/2} \cos(\omega_a t)\Theta(-t)$ and $e^{\kappa_b t/2} \cos(\omega_b t)\Theta(-t)$ respectively, where $\omega_{a,b}, \kappa_{a,b}$ are defined below. The specific temporal mode profile can be implemented using a Q-switch [82, 83] and is necessary for these modes to be subsequently captured in resonators for the joint-photon-number-modulo-2 measurement. The total state of the system, comprising of Alice, Bob, arnie and bert, can be written...
as:

\[
|\Psi_{ABab}\rangle = \frac{1}{2} \left( |g, g, C_{\alpha}^+, C_{\alpha}^+\rangle + |e, e, C_{\alpha}^-, C_{\alpha}^-\rangle + |g, e, C_{\alpha}^+, C_{\alpha}^-\rangle + |e, g, C_{\alpha}^-, C_{\alpha}^+\rangle \right) \quad (5.3)
\]

Next, a joint-photon-number-modulo-2 measurement is performed on these propagating microwave modes as follows. The propagating microwave modes, entangled with the stationary qubits, pass through transmission lines and are resonantly incident on two cavities, exciting their fundamental modes with frequencies (decay rates) \(\omega_a(\kappa_a)\) and \(\omega_b(\kappa_b)\), respectively. Due to the specific form of the chosen mode-profile, at \(t = 0\), these propagating modes get perfectly captured in these cavities. Subsequently, their joint-photon-number-modulo-2 is measured by coupling a transmon qubit to these modes [79]. An even (odd) joint-photon-number-modulo-2 outcome corresponds to a measurement result \(p = +1(p = -1)\) and the four-mode state, in absence of transmission losses and measurement imperfections, can be written as:

\[
|\Psi_{ABab}^{p=1}\rangle = \frac{1}{\sqrt{2}} \left( |g, g, C_{\alpha}^+, C_{\alpha}^+\rangle + |e, e, C_{\alpha}^-, C_{\alpha}^-\rangle \right), \quad (5.4)
\]

\[
|\Psi_{ABab}^{p=-1}\rangle = \frac{1}{\sqrt{2}} \left( |g, e, C_{\alpha}^+, C_{\alpha}^-\rangle + |e, g, C_{\alpha}^-, C_{\alpha}^+\rangle \right). \quad (5.5)
\]

After this measurement, the ancilla qubits, arnie and bert, are entangled to Alice and Bob. The last step of the protocol performs the crucial function of disentangling Alice and Bob from the propagating ancillas, while preserving the entanglement between Alice and Bob. This is done by performing homodyne measurement along the direction \(\text{arg}(\alpha)\) of each of the outgoing microwave modes arnie and bert after the joint-photon-number-modulo-2 measurement. Since we have chosen \(\alpha \in \mathbb{R}\), the X-quadratures of the microwave modes need to be measured. From Eqs. (5.4), (5.5), it is evident that the pair of outcomes of the integrated homodyne signal \((x_a, x_b)\) in the vicinity of \((\alpha, \alpha)\) or \((-\alpha, -\alpha)\) projects Alice and Bob to \(|+\rangle_p |+\rangle_p / \sqrt{2}\). Similarly, an outcome in the vicinity of \((\alpha, -\alpha)\) or \((-\alpha, \alpha)\) projects Alice and Bob to \(|+\rangle_p |-\rangle_p / \sqrt{2}\).

For a given \(p\), after the homodyne detections of arnie and bert, the density matrix \(\rho_{ABab}^p = |\Psi_{ABab}^p\rangle \langle \Psi_{ABab}^p|\) evolves to:

\[
\rho_{ABab}^p \rightarrow \frac{\mathcal{M}_X \rho_{ABab}^p \mathcal{M}_X^\dagger}{\text{Tr}[\mathcal{M}_X \rho_{ABab}^p \mathcal{M}_X^\dagger]}, \quad \mathcal{M}_X = |x_a, x_b\rangle \langle x_a, x_b|.
\]

The probability distribution of the outcomes \(P^p(x_a, x_b) = \frac{1}{2} \text{Tr}[\mathcal{M}_X \rho_{ABab}^p \mathcal{M}_X^\dagger]\). The factor of \(1/2\) arises because each of the outcomes \(p = \pm 1\) occurs with \(1/2\) probability. The resulting state of Alice
Figure 5.3: Probability distribution $P^p(x_a, x_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix $\rho_{AB}$ to the Bell-states $|\phi^\pm\rangle = (|g,g\rangle \pm |e,e\rangle)/\sqrt{2}$ are shown for the case when the joint-photon-number-modulo-2 measurement yields $p = 1$. We choose $\alpha = 1$ and assume absence of measurement imperfections and photon loss. (Left) Probability distribution showing four Gaussian distributions centered at $x_a = \pm \alpha, x_b = \pm \alpha$. (Center and Right) Corresponding overlap to the Bell-state $|\phi^+\rangle$ tends to 1 for $(x_a, x_b)$ in the vicinity of $(\alpha, \alpha)$ and $(-\alpha, -\alpha)$. Similarly, overlap to the Bell-state $|\phi^-\rangle$ tends to 1 for $(x_a, x_b)$ in the vicinity of $(-\alpha, \alpha)$ and $(\alpha, -\alpha)$. For an outcome on one of the lines: $x_a = 0$ or $x_b = 0$, the resultant state of Alice and Bob is an equal superposition of $|\phi^+\rangle$ and $|\phi^-\rangle$ and is not an entangled state. For $p = -1$, identical results are obtained with the substitution: $|\phi^\pm\rangle \rightarrow |\psi^\pm\rangle$.

and Bob is obtained by tracing out the states of arnie and bert. We evaluate $P^p(x_a, x_b)$ and the corresponding overlap to the Bell-states $|\phi^\pm\rangle = (|g,g\rangle \pm |e,e\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|g,e\rangle \pm |e,g\rangle)/\sqrt{2}$ to be (see Appendix F.2.1):

$$P^p(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2 + x_b^2)} e^{-4\alpha^2 N_p},$$

$$\langle \phi^+ | \rho_{AB}^p | \phi^+ \rangle = \frac{1 + p}{2} \left[ \frac{1}{2} \pm \frac{\sinh(4x_a \alpha) \sinh(4x_b \alpha)}{4N_p (1 - e^{-4\alpha^2})} \right],$$

$$\langle \psi^+ | \rho_{AB}^p | \psi^+ \rangle = \frac{1 - p}{2} \left[ \frac{1}{2} \pm \frac{\sinh(4x_a \alpha) \sinh(4x_b \alpha)}{4N_p (1 - e^{-4\alpha^2})} \right],$$

where

$$N_{p=1} = \frac{\cosh^2(2x_a \alpha) \cosh^2(2x_b \alpha)}{(1 + e^{-2\alpha^2})^2} + \frac{\sinh^2(2x_a \alpha) \sinh^2(2x_b \alpha)}{(1 - e^{-2\alpha^2})^2},$$

$$N_{p=-1} = \frac{1}{1 - e^{-4\alpha^2}} \left[ \cosh^2(2x_a \alpha) \sinh^2(2x_b \alpha) + \sinh^2(2x_a \alpha) \cosh^2(2x_b \alpha) \right].$$

Fig. 5.3 shows the probability distribution $P^p(x_a, x_b)$ of the outcomes of the integrated homo-
dyne currents $x_a, x_b$, together with the overlap to the Bell-states $|\phi^+\rangle, |\phi^-\rangle$ for the case when the joint-photon-number-modulo-2-measurement outcome is $p = 1$. We choose $\alpha = 1$ in absence of transmission loss and measurement inefficiency. The probability distribution [Eq. (5.7)] contains four Gaussian distributions centered around $x_a = \pm \alpha, x_b = \pm \alpha$. For $(x_a, x_b)$ in the vicinity of $(\alpha, \alpha)$ and $(-\alpha, -\alpha)$, the overlap to the Bell-state $|\phi^+\rangle$ approaches 1, while for $(x_a, x_b)$ in the vicinity of $(\alpha, -\alpha)$ and $(-\alpha, \alpha)$, the overlap to the Bell-state $|\phi^-\rangle$ approaches 1. For outcomes along the lines $x_a = 0$ and $x_b = 0$, Alice and Bob are projected on to equal superpositions of $|\phi^+\rangle$ and $|\phi^-\rangle$ and thus, are not entangled. The results for the case $p = -1$ are identical with $|\phi^\pm\rangle$ is replaced by $|\psi^\pm\rangle$.

### 5.3.2 Implementation using mod 4 cat states

In this section, we describe the implementation of our protocol where the ancilla qubits are mapped on to mod 4 cat states. The first step of the protocol again involves generating entanglement between the stationary qubit of Alice (Bob) and the propagating microwave mode arnie (bert) giving rise to the following states: $(|g, g, C_{\alpha}^{0 \text{mod} 4} \rangle + |e, e, C_{\alpha}^{2 \text{mod} 4} \rangle)/\sqrt{2}$ $(|g, C_{\alpha}^{0 \text{mod} 4} \rangle + |e, C_{\alpha}^{2 \text{mod} 4} \rangle)/\sqrt{2})$. This set of entangled states can be obtained in an analogous manner using the method described in the previous subsection. We will again choose, without loss of generality, $\alpha, \beta \in \mathbb{R}, \alpha = \beta > 0$. The total state of the system, comprising of Alice, Bob, arnie and bert, can be written as:

$$
|\Psi_{ABab}\rangle = \frac{1}{2} (|g, g, C_{\alpha}^{0 \text{mod} 4} \rangle + |e, e, C_{\alpha}^{2 \text{mod} 4} \rangle + |g, e, C_{\alpha}^{0 \text{mod} 4} \rangle, C_{\alpha}^{2 \text{mod} 4} \rangle + |e, g, C_{\alpha}^{2 \text{mod} 4} \rangle, C_{\alpha}^{0 \text{mod} 4} \rangle).
$$

By suitably engineering the temporal mode profiles of the propagating modes as in the previous subsection, these propagating entangled qubit-photon states are then captured in resonators. Subsequently, a joint-photon-number-modulo-4 measurement is performed on these captured modes (see Appendix. F.1 for details of the measurement protocol). In absence of measurement imperfections and losses, the joint-photon-number-modulo-4 has two possible outcomes: $\lambda = 0, 2$ (the two-qubit measurement outcome $p$ of Sec. 5.2 can be written as $p = i^\lambda$), corresponding to which the four-mode state can be written as:

1. $|\Psi_{ABab}^{\lambda=0}\rangle = \frac{1}{\sqrt{2}} (|g, g, C_{\alpha}^{0 \text{mod} 4} \rangle, C_{\alpha}^{0 \text{mod} 4} \rangle + |e, e, C_{\alpha}^{2 \text{mod} 4} \rangle, C_{\alpha}^{2 \text{mod} 4} \rangle)$,

2. $|\Psi_{ABab}^{\lambda=2}\rangle = \frac{1}{\sqrt{2}} (|g, e, C_{\alpha}^{0 \text{mod} 4} \rangle, C_{\alpha}^{2 \text{mod} 4} \rangle + |e, g, C_{\alpha}^{2 \text{mod} 4} \rangle, C_{\alpha}^{0 \text{mod} 4} \rangle)$.
The final step of the protocol comprises of making homodyne detections of arnie and bert and here we choose the X-quadrature of both these modes. Similar calculations can be done for other choices. Consider the case when \( \lambda = 0 \). From Eq. (5.12), it follows that each homodyne detector will have Gaussian distributions centered around \( x, \lambda \). It also follows from Eq. (5.12) that for events \((x_a, x_b)\) in the vicinity of \((\pm \alpha, \pm \alpha)\) and \((0, 0)\), the resulting state of Alice and Bob is \(|\phi^+\rangle\), while for outcomes in the vicinity of \((0, \pm \alpha)\) and \((\pm \alpha, 0)\), the resulting state of Alice and Bob is \(|\phi^-\rangle\). Similar set of reasoning holds for \( \lambda = 2 \), when the states \(|\psi^\pm\rangle\) are generated. Since the state of Alice and Bob depend only on \((|x_a|, |x_b|)\), the resulting overlap distributions respect a four-fold rotational symmetry in the \((x_a, x_b)\) space (see Fig. 5.4).

After the homodyne detection, the density matrix of Alice, Bob, arnie and bert evolves to:

\[
\rho_{ABab}^\lambda \rightarrow \frac{\mathcal{M}_X \rho_{ABab}^\lambda \mathcal{M}_X^\dagger}{\text{Tr}[\mathcal{M}_X \rho_{ABab}^\lambda \mathcal{M}_X]}, \quad \mathcal{M}_X = |x_a, x_b\rangle\langle x_a, x_b|.
\]

(5.14)

The probability distribution of the outcomes \(P^\lambda(x_a, x_b) = \frac{1}{2\pi} \text{Tr}[\mathcal{M}_X \rho^\lambda \mathcal{M}_X]\). Subsequent state of Alice and Bob can again be obtained by tracing out the modes arnie and bert. Note again the factor of 1/2 in the expression for probability distribution due to the probability 1/2 of occurrence of either \( \lambda = 0 \) or \( \lambda = 2 \). Computing the probability of outcomes and the overlap to the Bell-states (see Appendix. F.2.2 for details), we arrive at:

\[
P^\lambda(x_a, x_b) = \frac{1}{2\pi} \frac{e^{-2(x_a^2 + x_b^2)}}{(1 + e^{-2\alpha^2})^2} \tilde{N}_\lambda,
\]

(5.15)

\[
\langle \phi^+ | \rho_{AB}^\lambda | \phi^+ \rangle = \frac{1 + i\lambda}{2} \left[ 1 \pm \frac{\prod_{\delta=0,2} \frac{F_\delta(x_a)F_\delta(x_b)}{\tilde{N}_\lambda}}{\prod_{\delta=0,2} \frac{\cosh(\alpha^2)}{\cosh(\alpha^2)}} \right],
\]

\[
\langle \psi^\pm | \rho_{AB}^\lambda | \psi^\pm \rangle = \frac{1 - i\lambda}{2} \left[ 1 \pm \frac{\prod_{\delta=0,2} \frac{F_\delta(x_a)F_\delta(x_b)}{\tilde{N}_\lambda}}{\prod_{\delta=0,2} \frac{\cosh(\alpha^2)}{\cosh(\alpha^2)}} \right],
\]

where

\[
\tilde{N}_\lambda = \frac{F_0(x_a)F_0(x_b)}{1 + \frac{\cosh(\alpha^2)}{\cosh(\alpha^2)}}, \quad \tilde{N}_\lambda = \frac{1}{1 - \{\frac{\cosh(\alpha^2)}{\cosh(\alpha^2)}\}^2} \left[ F_0(x_a)^2F_2(x_b)^2 + F_2(x_a)^2F_0(x_b)^2 \right],
\]

(5.16)

(5.17)

and \(F_\lambda(x) = e^{-\alpha^2 \cosh(2\alpha x)} + i\lambda \cos(2\alpha x)\).

Fig. 5.4 shows the probability of success and the overlap to the Bell-states \(|\phi^\pm\rangle, |\psi^\pm\rangle\) for this
Figure 5.4: Probability distribution $P^\lambda(x_a, x_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix $\rho_{AB}$ with the Bell-states $|\phi^\pm\rangle = (|g, g\rangle \pm |e, e\rangle)/\sqrt{2}$, $|\psi^\pm\rangle = (|g, e\rangle \pm |e, g\rangle)/\sqrt{2}$ are shown. We choose $\alpha = 1$ in absence of measurement imperfections and photon losses. The top (bottom) left panel shows the probability of outcomes for the joint-photon-number-modulo-4 outcome $\lambda = 0(2)$. Corresponding overlaps to the Bell-states $|\phi^\pm\rangle(|\psi^\pm\rangle)$ are plotted in the top (bottom) center and top (bottom) right panels. The overlaps to the odd (even) Bell-states for $\lambda = 0(2)$ are zero and not shown for brevity. For both $\lambda = 0$ and $\lambda = 2$, one gets entangled Bell-states for Alice and Bob for majority of outcomes in the $(x_a, x_b)$-space. The alternating bright and dark fringes in plots are due to the measurement of X-quadrature of both arnie and bert, both of which are in superpositions of $|C_+^\alpha\rangle$ and $|C_{\alpha \alpha}^+\rangle$. The size of the fringes decreases with increasing values of $\alpha$. 
implementation of our protocol. We choose $\alpha = 1$, and plot the results for the two possible joint-photon-number-modulo-4 measurement outcomes: $\lambda = 0, 2$, in absence of imperfections and photon losses. For a majority of outcomes in the $(x_a, x_b)$-space, we get one of the four aforementioned Bell-states. The existence of fringes in the plots is due to the measurement of the X-quadratures of both arnie and bert, each of which are in superpositions of $|C^{+}_\alpha\rangle$ and $|C^{-}_\alpha\rangle$. The size of the fringes decreases with increasing $\alpha$. The overlaps to the odd (even) Bell-states for the case $\lambda = 0(2)$ are identically equal to zero.

So far, we have described our protocol in absence of measurement imperfections and propagation losses. In what follows, we incorporate measurement inefficiencies and propagating losses in our computational model and investigate the resilience of the two different implementations of our protocol to these imperfections.

5.4 Finite quantum efficiency and non-zero photon loss

The dominant source of imperfections in current circuit-QED systems that affect our protocol is undesired photon loss. These losses occur due to photon attenuation on the transmission lines and other lossy devices like circulators and isolators which are necessary for an actual experimental implementation. These lead to decoherence of the entangled states of Alice (Bob) and arnie (bert) as arnie and bert propagate from the stationary qubits to the joint-photon-number-modulo-2/4 measurement apparatus and from there on to the homodyne detectors. For simplicity, we assume the losses to be equal for both arnie and bert (the case of unequal losses can be calculated easily using the method described below). Thus, for each of arnie and bert, we define two efficiency parameters $\eta_1$ and $\eta_2$. Here, $\eta_1$ models the losses before the joint-photon-number-modulo-2/4 measurement apparatus and $\eta_2$ models the losses thereafter and before homodyne detection setup. These losses are modeled as photons lost by each of arnie and bert as they pass through beam-splitters with transmission probabilities $\eta_1$ and $\eta_2$ in otherwise perfect transmission lines [102]. Finite qubit lifetimes, with current circuit-QED system parameters, are much less dominant source of imperfection compared to photon loss and are thus neglected in the subsequent analysis.

First, we qualitatively describe the effect of undesired single-photon losses for the two implementations. Consider the case when the ancilla qubits are encoded in Schrödinger cat states. Loss of a photon is a bit-flip error on the ancilla qubit since $a(C^{\pm}_\alpha) \approx \alpha(C^{\mp}_\alpha)$, where $a$ is the annihilation operator of the propagating temporal mode. This bit-flip error occurs randomly as the entangled qubit-photon states of Alice (Bob) and arnie (bert) propagate from the stationary qubits to the
joint-photon-number-modulo-2/4 measurement apparatus and from thereon to the homodyne detectors. This results in decoherence of the entangled states of Alice (Bob) and arnie (bert). Therefore, the probability of generating a high fidelity Bell-state of Alice and Bob diminishes drastically (see Sec. 5.4.1).

Now, consider the case when the ancillas are encoded in the mod 4 cat states. To lowest order in photon loss, either arnie or bert can lose a photon. On losing a photon, the state of arnie or bert goes from $|C_0(2)\mod 4\alpha\rangle$ to the state $|C_3(1)\mod 4\alpha\rangle$. Therefore, when either arnie or bert loses a photon, the joint-photon-number-modulo-4 measurement now yields the values $\lambda = 1$ or $3$, unlike the perfect case outcomes $\lambda = 0$ or $2$ [see Eqs. (5.12), (5.13) and Sec. 5.4.2]. Thus, measurement of the joint-photon-number-modulo-4 allows us to keep track of loss of a photon in arnie or bert. This tracking of a single photon loss error is equivalent to correcting this error since it allows the knowledge of exact state of the four-qubits after the error has happened. As will be shown below, this enables generation of high fidelity entangled states of Alice and Bob with higher success rates than the mod 2 implementation.

In the next order in photon loss, either both arnie and bert lose one photon each or arnie loses two photons, or bert loses two photons. Consider the case in the mod 4 encoding when arnie and bert each lose a photon. Now, the measurement outcome $\lambda$ can be 0 or 2 as in the perfect case and just a measurement of $\lambda$ does not reveal if arnie and bert have indeed lost a photon each. However, these events of loss of one photon each in the ancillas can be tracked by individual photon-number-modulo-2 measurements of the ancillas. In this way, we can suppress the loss of coherence due to loss of a photon in both arnie and bert. Note that the other second order or the higher order losses cannot be suppressed by this encoding (see Sec. 5.5).

In next two subsections, we describe the effect of single-photon losses on the the two implementations of our protocol. This is followed by a comparison of the two. Lastly, we describe the protocol with added individual photon-number-modulo-2 measurements of arnie and bert.

### 5.4.1 Implementation using Schrödinger cat states

Consider the case when the ancilla qubits are encoded in Schrödinger cat states (see Sec. 5.3.1). First, we describe the calculation of the state of Alice and arnie after the propagation of arnie through the transmission line in presence of imperfections. The state of Bob and bert can be computed in an analogous manner. Following local entanglement generation between Alice and arnie, their states
can be written as:

\[ |\Psi_{Aa}\rangle = \frac{1}{\sqrt{2}} (|g, C_{\alpha}^{+}\rangle + |e, C_{\alpha}^{-}\rangle) \]

\[ = \frac{1}{\sqrt{2}} \sum_{j,\mu=0}^{1} N_{j}(-1)^{j\mu} |j, (-1)^{\mu}\alpha\rangle, \quad (5.18) \]

where \( N_{j} = 1/\sqrt{2(1 + (-1)^{j} e^{-2|\alpha|^2})} \Rightarrow N_{0(1)} = N_{+(-)} \) and \(|j = 0(1)\rangle = |g(e)\rangle\). To compute the decoherence due to propagation losses, without loss of generality, one can introduce an auxiliary propagating mode \( a' \), initialized in vacuum, and pass the joint-states of Alice, arnie and \( a' \) through a beam-splitter with transmission probability \( \eta_{1} \). Subsequent tracing out of the auxiliary mode yields the reduced density matrix for Alice and arnie after the entangled states of Alice and arnie propagated along the transmission line and arrived at the joint-photon-number-modulo-2 apparatus.

Therefore, just prior to making the joint-photon-number-modulo-2 measurement, we can write the total state of the four-modes to be (cf. Appendix F.3.1 for details of the calculation):

\[ \rho_{ABab} = \frac{1}{64} \sum_{Aa,Bb} \frac{N_{j}N_{j}'}{N_{k}N_{k'}N_{m}N_{m'}} (-1)^{\mu \cdot (j+k)+\nu \cdot (l+m)} e^{-\epsilon^{2} (2(-1)^{\nu + \nu'}(-1)^{\nu'})} (|j, l\rangle\langle j', l'|) \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad |C_{\alpha}^{(-)k}, C_{\alpha}^{(-)m}\rangle \langle C_{\alpha}^{(-)k'}, C_{\alpha}^{(-)m'}|, \quad (5.19) \]

where \( \sum_{Aa} = \sum_{j,j'=0}^{1} \sum_{k,k'=0}^{1} \sum_{\mu,\mu'=0}^{1} \sum_{l,l'=0}^{1} \sum_{m,m'=0}^{1} \sum_{\nu,\nu'=0}^{1} \mu = \{\mu,\mu'\}, \nu = \{\nu,\nu'\}, j = \{j, j'\}, k = \{k, k'\}, l = \{l, l'\}, m = \{m, m'\} \). Furthermore, we have defined \( \tilde{N}_{j} = 1/\sqrt{2(1 + (-1)^{j} e^{-2|\alpha|^2})} \) where \( \tilde{\alpha} = \sqrt{\eta_{1} \alpha}, \epsilon = \sqrt{1 - \eta_{1} \alpha} \).

Note that Eq. (5.19) is expressed in the eigenbasis of the joint-photon-number-modulo-2 measurement. An outcome of \( p = 1(-1) \) results in the state of Alice, Bob, arnie and bert to be in \( \rho_{ABab}' = \rho_{ABab}' / \text{Tr}[\rho_{ABab}'] \), where the post-measurement un-normalized density matrix \( \rho_{ABab}' \) is given by:

\[ \rho_{ABab}' = \frac{1}{64} \sum_{Aa,Bb} \frac{N_{j}N_{j}'}{N_{k}N_{k'}N_{m}N_{m'}} (-1)^{\mu \cdot (j+k)+\nu \cdot (l+m)} e^{-\epsilon^{2} (2(-1)^{\nu + \nu'}(-1)^{\nu'})} (|j, l\rangle\langle j', l'|) \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad |C_{\alpha}^{(-)k}, C_{\alpha}^{(-)m}\rangle \langle C_{\alpha}^{(-)k'}, C_{\alpha}^{(-)m'}|, \quad (5.20) \]

Here the prime in \( \sum_{Aa,Bb}^{'} \) indicates that the summation has to be performed while keeping \( k + m = 0(1) \text{ mod } 2, k' + m' = 0(1) \text{ mod } 2 \) for an outcome of \( p = 1(-1) \).

Next, we treat the photon-losses after the joint-photon-number-modulo-2 measurement setup and before the homodyne detectors. We model the measurement operator for the imperfect homodyne
detection with efficiency $\eta_2$ as a superposition of projectors $|x_a, x_b\rangle \langle x_a, x_b|$ with a Gaussian envelope [102, 103]:

$$\Omega_Q = \frac{1}{\sigma \sqrt{\pi \eta_2}} \int_{-\infty}^{\infty} dx_a \int_{-\infty}^{\infty} dx_b \, e^{-\frac{1}{2\eta_2^2} \left(\frac{x_a}{\sqrt{\eta_2}} - x_a\right)^2} e^{-\frac{1}{2\sigma^2} \left(\frac{\eta_2}{\sqrt{\sigma^2}} - x_b\right)^2} |x_a, x_b\rangle \langle x_a, x_b|,$$

$$\sigma^2 = \frac{1 - \eta_2}{2\eta_2}. \quad (5.21)$$

Here $q_{a(b)}$ denote the imperfect measurement outcome which, in principle, can arise out of the possible perfect measurement outcomes $x_{a(b)}$ with a Gaussian probability distribution shown above. For $p = \pm 1$, the system density matrix due to this measurement evolves as:

$$\rho^{p}_{ABab} \rightarrow \Omega_Q \rho^{p}_{ABab} \Omega^\dagger_Q \frac{\sigma_{ABab}^{p}}{\text{Tr}[\Omega_Q \rho^{p}_{ABab} \Omega^\dagger_Q]}, \quad (5.22)$$

where the product $\text{Tr}[\rho^{p}_{ABab}] \times \text{Tr}[\Omega_Q \rho^{p}_{ABab} \Omega^\dagger_Q]$ is the probability distribution of outcomes $\bar{P}^p(q_a, q_b)$. These can be evaluated analytically following the method outlined in Appendix. F.2.1, along with the overlap to the Bell-states $|\phi^\pm\rangle, |\psi^\pm\rangle$. The explicit forms of these quantities are not provided for the sake of brevity.

Fig. 5.5 shows the probability distribution of outcomes for the imperfect measurement $\bar{P}^p(q_a, q_b)$ and the overlaps to the Bell-states $|\phi^\pm\rangle$ for the case when the joint-photon-number-modulo-2 measurement yields $p = 1$. We choose $\alpha = 1$ and the measurement inefficiencies to be: $\eta_1 = \eta_2 = 0.8$. The corresponding distribution shows four Gaussian distributions centered at $q_a = \pm \bar{\alpha}, q_b = \pm \bar{\alpha}$. The corresponding overlap to the Bell-state $|\phi^+\rangle$ is substantial for $(q_a, q_b)$ in the vicinity of $(\bar{\alpha}, \bar{\alpha})$ and $(-\bar{\alpha}, -\bar{\alpha})$, while the overlap to the Bell-state $|\phi^-\rangle$ is substantial for $(q_a, q_b)$ in the vicinity of $(\bar{\alpha}, -\bar{\alpha})$ and $(-\bar{\alpha}, \bar{\alpha})$. Note that in presence of imperfections, the maximum fidelity for outcomes with non-negligible occurrence probability is $\sim 0.7$ instead of 1.0 computed earlier for the case of perfect efficiency (compare Fig. 5.3). For outcomes along the lines $q_a = 0$ and $q_b = 0$, one does not generate entangled states for reasons similar to the case of perfect efficiency. Similar set of results hold for $p = -1$.

### 5.4.2 Implementation using mod 4 cat states

In this section, we consider the case when the ancilla qubits are encoded in the mod 4 cat states discussed in Sec. 5.3.2. The computation for this case is, in principle, similar to that outlined in the previous subsection. We begin by considering the entangled qubit-photon state of Alice and arnie,
Figure 5.5: Probability distribution $\tilde{P}^p(q_a, q_b)$ of outcomes of the homodyne measurements of Arnie and Bert and resulting overlaps of Alice and Bob’s joint density matrix $\rho_{AB}$ to the Bell states $|\phi^\pm\rangle$ are shown for the case when the joint-photon-number-modulo-2 measurement yields $p = 1$. We choose $\alpha = 1$ and $\eta_1 = \eta_2 = 0.8$. (Left) Probability distribution showing four Gaussian distributions centered at $q_a = \pm \bar{\alpha}, q_b = \pm \bar{\alpha}$. (Center and Right) The corresponding overlap to the Bell-state $|\phi^+\rangle$ is substantial for $(q_a, q_b)$ in the vicinity of $(\bar{\alpha}, \bar{\alpha})$ and $(-\bar{\alpha}, -\bar{\alpha})$, while the overlap to the Bell-state $|\phi^-\rangle$ is substantial for $(q_a, q_b)$ in the vicinity of $(\bar{\alpha}, -\bar{\alpha})$ and $(-\bar{\alpha}, \bar{\alpha})$. We note that the maximum fidelity Bell-state that can be obtained for outcomes with non-negligible occurrence probability is $\sim 0.7$ instead of 1.0 that was obtained for perfect efficiency (compare Fig. 5.3). Outcomes along the lines $q_a = 0$ and $q_b = 0$ do not yield entangled states for reasons similar to that given for perfect efficiency. Similar results hold for the case $p = -1$. 
which, in this case, is given by:

\[
|\Psi_{\lambda}\rangle = \frac{1}{\sqrt{2}}(|g, C^{0 \mod 4}_\alpha \rangle + |e, C^{2 \mod 4}_\alpha \rangle)
\]

\[
= \frac{1}{\sqrt{2}} \sum_{j, \mu, \nu = 0}^{1} \mathcal{N}_{2j} \mathcal{N}_{2j} (-1)^{2\nu} |j, (-1)^\mu i^\nu \alpha\rangle,
\]

(5.23)

where \(\mathcal{N}_{2j}\) and the states \(|j = 0, 1\rangle\) for Alice are defined as before. The definition of \(\tilde{\mathcal{N}}_{2j}\) is obtained by setting \(\lambda = 2j\) in the following:

\[
\tilde{\mathcal{N}}_\lambda = \left[2 + 2(-i)^\lambda \{e^{i|\alpha|^2} + (-1)^\lambda e^{-i|\alpha|^2}\} / \{e^{i|\alpha|^2} + (-1)^\lambda e^{-i|\alpha|^2}\} \right]^{-\frac{1}{2}},
\]

\(\lambda = \{0, 1, 2, 3\}\). To compute the state of Alice and Arnie after their entangled qubit-photon state encounter propagation losses, we follow a similar approach as in the previous section: introducing an auxiliary mode \(\alpha'\), looking at the resultant state after passage through a beam-splitter with transmission probability \(\eta_1\) and subsequently, tracing out the mode \(\alpha'\). Similar set of calculations can be done for Bob and Bert. Performing this computation results in (cf. Appendix F.3.2 for details):

\[
\rho_{ABab} = \frac{1}{2^{10}} \sum_{\alpha a, \beta b} \prod_j \mathcal{N}_{2\lambda} \mathcal{N}_{2\lambda} (-1)^{\mu' + \nu + \phi + \psi + k} e^{-\frac{1}{2} \left[2 - (-1)^{\mu + \phi} - (-1)^{\nu + \psi} - (-1)^{\nu' + \psi'}\right]}
\]

\[
\times \langle j, k | j', k' \rangle \left(\mathcal{G}^{\gamma \mod 4}_{\alpha}, \mathcal{G}^{\delta \mod 4}_{\alpha} \right) \langle C^{\gamma' \mod 4}_{\alpha}, C^{\delta' \mod 4}_{\alpha} \rangle
\]

(5.24)

where \(\prod_j \prod_{\lambda} \sum_{\alpha} \sum_{\mu, \nu, \phi, \psi, k, k'} \sum_{\gamma, \gamma', \delta, \delta'} \sum_{\lambda} \sum_{\delta} \sum_{\lambda} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta} \sum_{\delta} \sum_{\gamma} \sum_{\gamma} \sum_{\delta}

and \(j = \{j, j', \mu, \nu, \phi\}, \nu = \{\nu, \nu'\}\) and \(\gamma = \{\gamma, \gamma', \delta, \delta'\}\), \(\lambda = \{\delta, \delta'\}\). The definitions of \(\bar{\gamma}, \bar{\gamma}\) can obtained from the definitions of \(\gamma, \gamma\) (cf. Secs. 5.4.1, 5.4.2) by making the substitution \(\alpha \rightarrow \bar{\alpha}\) and \(\bar{\alpha}, \bar{\epsilon}\) have been defined in the previous subsection.

Noting that Eq. (5.24) expresses the density matrix in the eigenbasis of the joint-photon-number-modulo-4 measurement of Arnie and Bert, an outcome \(\lambda \in \{0, 1, 2, 3\}\) projects the state of the four modes to \(\rho_{ABab}^\lambda = \rho_{ABab}^\lambda / \text{Tr} \rho_{ABab}^\lambda\), where \(\rho_{ABab}^\lambda\) is the un-normalized density matrix obtained from \(\rho_{ABab}\) by restricting the summation of \(\gamma, \gamma', \delta, \delta'\) to be such that: \(\gamma + \delta - \lambda \mod 4, \gamma' + \delta' - \lambda \mod 4\). The inefficiencies in the final homodyne detection of Arnie and Bert can be done similarly to the method described in Sec. 5.4.1, using Eq. (5.21). The probability of outcomes and the overlap to the Bell-states \(|\phi^\pm\rangle, |\psi^\pm\rangle\) can be evaluated analytically, whose explicit forms are not shown here for brevity.

Fig. 5.6 shows the probability of outcomes and the overlap to the Bell-states \(|\phi^\pm\rangle, |\psi^\pm\rangle\) when the joint-photon-number-modulo-4 measurement outcome \(\lambda = 0(2)\) (see Fig. F.1 in Appendix. F.3.2
Figure 5.6: Probability distribution $P^\lambda (q_a, q_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlaps of Alice and Bob’s joint density matrix $\rho_{AB}$ with the Bell-states $|\phi^\pm \rangle = (|g, g\rangle \pm |e, e\rangle)/\sqrt{2}$, $|\psi^\pm \rangle = (|g, e\rangle \pm |e, g\rangle)/\sqrt{2}$ are shown. We choose $\alpha = 1.0$ and $\eta_1 = \eta_2 = 0.8$ and show the cases $\lambda = 0, 2$ (see Fig. F.1 in Appendix. F.3.2 for $\lambda = 1, 3$). The top (bottom) left panel shows the probability of outcomes for the joint-photon-numbermodulo-4 outcome to be $\lambda = 0(2)$. Corresponding overlaps to the Bell-states $|\phi^\pm \rangle (|\psi^\pm \rangle)$ are plotted in the top (bottom) center and top (bottom) right panels. Note that the maximum fidelity obtained for outcomes with non-negligible occurrence probability is lowered compared to the perfect case (compare Fig. 5.4). The overlap to the odd (even) Bell-states for $\lambda = 0(2)$ are not shown for brevity.
for $\lambda = 1, 3$). We have chosen $\alpha = 1.0$ and $\eta_1 = \eta_2 = 0.8$. The top (bottom) left panel shows the probability of outcomes for $\lambda = 0(2)$, while the top (bottom) center and right panels show the corresponding overlaps to the Bell-states $|\phi^\pm\rangle (|\psi^\pm\rangle)$. We see that including inefficiencies lowers the maximum fidelity obtained for outcomes with non-negligible occurrence probability compared to perfect case (compare Fig. 5.4).

5.4.3 Comparison of the mod 2 and mod 4 implementations in presence and absence of imperfections

In the previous subsections, we described the probability of outcomes for the two different implementations of our protocol. In this section, we discuss the total and optimized success-rates of generating entangled states for the two implementations, comparing the cases of perfect and imperfect quantum efficiencies.

First, consider the mod 2 implementation of our protocol for perfect and imperfect quantum efficiencies. The probability of success and the overlaps to the Bell-states are given in Fig. 5.3 (Fig. 5.5) for the perfect (imperfect) case. The total success-rate for generation of entangled states can be computed for different cut-off fidelities by integrating the appropriate region of $(x_a, x_b)$ or $(q_a, q_b)$-space of outcomes. In the perfect (imperfect) case, the majority of the outcomes, occurring around $\pm \alpha (\pm \bar{\alpha})$, give rise to entangled states, while the events along the lines $x_a(q_a) = 0$ and $x_b(q_b) = 0$ do not. Thus, in order to have a high total success-rate of generating entangled states, the number of outcomes along the lines $x_a(q_a) = 0$ and $x_b(q_b) = 0$ should be minimized. This can be done by increasing the size of $\alpha$ because the probability of obtaining an outcome along these lines goes down exponentially with $\alpha^2 (\bar{\alpha}^2)$. While in the perfect case $\alpha$ can be made arbitrarily large giving rise to deterministic generation of entangled states, in the imperfect case, too large a value of $\alpha$ lowers the success-rate. This is because large values of $\alpha$ are more susceptible to photon-losses.

Next, consider the mod 4 implementation of our protocol. The relevant probability of outcomes and overlap to the Bell-states are given in Fig. 5.4 (Fig. 5.6) for the perfect (imperfect) case. Similar considerations, as in the mod 2 implementation, lead to the conclusion that in absence of imperfections, increasing $\alpha$, in general, increases the success-rate for the different cut-off fidelities. Note that the increase, however, is not monotonic (see below). However, in presence of imperfections, arbitrarily increasing $\alpha$ does not increase the success-rate of generating high-fidelity entangled states. This happens again because large values of $\alpha$ are more susceptible to photon-losses.

Fig. 5.7 shows the total probability of generation of entangled states as a function of different
Figure 5.7: Total success probability ($P_{\text{total}}$) for different cut-off fidelities and different choices of $\alpha$ is plotted for the case of perfect quantum efficiency (left panels, indicated by $\eta_1 = \eta_2 = 1$) and the imperfect case (right panels, where we have chosen $\eta_1 = \eta_2 = 0.8$). The top (bottom) panels correspond to the mod 2 (mod 4) implementation. (Top left) For $\alpha \ll 1$, the probability of generation of entangled states with overlap $> 0.9$ is around 0.5. Increasing $\alpha$ to $\gg 1$ generates perfect entangled states with near-unit probability. (Top right) In presence of imperfections, for $\alpha \ll 1$, we generate entangled states with overlap $> 0.7$ to Bell-states with probability in excess of 0.3. However, increasing $\alpha$ does not lead to a higher success-rate for generating better entangled states. This is because larger values of $\alpha$ are more susceptible to photon-losses. Depending on the desired cut-off fidelity and the efficiency of an experimental setup, there is an optimal choice of $\alpha$ that leads to the maximal success-rate. For instance, in the case shown, for a desired cut-off fidelity of 0.75, the optimal choice is $\alpha \simeq 0.7$. (Bottom left) The total probability of generating entangled states with overlap $> 0.9$ is $\sim 0.3$ and increasing $\alpha$ increases the success-rate to near-unity. The increase is non-monotonic because the size of the fringes in the overlap (Figs. 5.4, 5.6) depend on the value of $\alpha$. (Bottom right) In presence of imperfections, increasing $\alpha$ to $\gg 1$ no longer increases the success-rate. As in the mod 2 case, there is an optimal choice for $\alpha$: e.g. for a cut-off fidelity of 0.75 for this choice of inefficiency, $\alpha \sim 1.5$. Note that for relatively large values of $\alpha \sim 1.5$, it is more advantageous to use the mod 4 implementation over the mod 2 implementation. This is because the mod 4 protocol corrects for the decoherence of the entangled qubit-photon states due to photon-loss to first order.
cut-off fidelities and different choices of the parameter $\alpha$ for the perfect case (left panels, indicated by $\eta_1 = \eta_2 = 1$) and the imperfect case (right panels, for which, we have chosen $\eta_1 = \eta_2 = 0.8$). The top (bottom) panels correspond to the mod 2 (mod 4) implementation. For the mod 2 implementation, for the perfect case, for $\alpha \ll 1$, the probability of generation of entangled states with an overlap $> 0.9$ to a Bell-state is $\sim 0.5$ while for $\alpha > 1$ for which we generate entangled states with unit-probability. On the other hand, in the imperfect case, for the choice of efficiency parameters $\eta_1 = \eta_2 = 0.8$, small values of $\alpha$ ($\alpha \ll 1$) give rise to entangled states with overlaps to Bell-states $> 0.7$ with a success-rate in excess of 0.3. However, unlike the perfect case, larger values of $\alpha$ do not help getting better success-rate for similar or better entangled states because of photon-losses. Thus, for different cut-off fidelities and measurement efficiencies, there is an optimal choice of $\alpha$, e.g. in the case shown, for a cut-off fidelity of 0.75, the optimal choice for $\alpha$ is $\sim 0.7$. For the mod 4 implementation, the success-rates are $\sim 0.3$ for generating entangled states with overlap $> 0.9$ in absence of imperfections for $\alpha \ll 1$. Increasing $\alpha$ increases the success-rate to near-unity. Note that the increase is non-monotonic, unlike the case of mod 2 implementation. This is because the size of the fringes, which are regions of unentangled states, present in the overlap to the Bell-states (see Figs. 5.4, 5.6) depends on the value of $\alpha$. In presence of imperfections, increasing $\alpha$ does not necessarily increase the success-rate due to the decoherence due to photon losses. Just as in the mod 2 implementation, there is an optimal choice for $\alpha$: e.g. for a cut-off fidelity of 0.75 for this choice of inefficiency, $\alpha \sim 1.5$. Note that for relatively larger values of $\alpha > 1.5$, it is more advantageous to use the mod 4 implementation over the mod 2 implementation since it corrects for the decoherence of the propagating qubit-photon states due to photon-loss to first order.

Next, we optimize the success-rate with respect to the parameter $\alpha$ for the two implementations. This optimization is done numerically for the different values of the efficiency parameters $\eta_1, \eta_2$ and the different cut-off fidelities. This is shown below in Fig. 5.8. We take $\eta_1 = \eta_2$ for simplicity. For $\eta_1 = \eta_2 = 0.8$, one is able to generate entangled states with overlap to Bell-states $\sim 0.75$ with a near-unity success-rate by both mod 2 and mod 4 implementation. Although the probability of generation of higher fidelity Bell-states decreases for both the implementations, the rate of decrease is different for the two. Enclosed by the white curves in the right panel is the region where the mod 4 implementation has a higher success rate than the mod 2 implementation. For instance, for $\eta_1 = \eta_2 = 0.9$, the probability of generating a Bell-state with overlap of 0.95 or greater is less than $10^{-10}$ for the mod 2 implementation (white rectangle in left panel). On the other hand, the mod 4 implementation is able to generate these states with a success-rate of $10^{-4}$ (white rectangle in right panel). This is because of its ability to correct for photon loss errors to first order. Note, however,
Figure 5.8: (Top panels) Optimized total success-probability $P^*$ in logarithmic scale for different cut-off fidelities and different choices of $\eta_1, \eta_2$ for the mod 2 and the mod 4 implementations. The optimization is done numerically for the value of the parameter $\alpha$. We choose, for simplicity, $\eta_1 = \eta_2$. The left (right) panels correspond to the joint-photon-number-modulo-2(4) implementation. For both protocols, for an efficiency of $\eta_1 = \eta_2 = 0.8$, entangled states with overlaps to the Bell-states $\sim 0.75$ are generated with near-unity success-rate. The probability of generating high-fidelity Bell-states diminishes rapidly. However, the rate of decrease of success-rate for the mod 2 and mod 4 implementations are different. The white curves in the right panel enclose the region for which the mod 4 implementation has a higher success rate than the mod 2 implementation. For instance, for inefficiency values $\sim 0.9$, the success-rate for the mod 2 implementation is less than $10^{-10}$ for generating states with overlap to Bell-states of $\sim 0.95$ (white rectangle in left panel). However, the mod 4 implementation, due to its ability to correct for photon loss errors to first order, can, in fact, generate states with overlap $\sim 0.95$ to Bell-states with a success-rate of $10^{-4}$ (white rectangle in right panel). However, the error correcting mod 4 protocol ceases to be advantageous to generate high fidelity Bell-states for low enough efficiencies and high enough cut-off fidelities. This is because for such low efficiency, higher order photon loss become more dominant.

that for low enough efficiency and high enough cut-off fidelity, the error correcting mod 4 protocol ceases to be advantageous. This is because for such low efficiency because higher than first order photon loss become more dominant.

5.4.4 Adding individual photon-number-modulo-2 measurements to the mod 4 implementation

In the previous subsections, we have shown that in presence of finite quantum efficiency, it is more advantageous to use the mod 4 implementation of our protocol, because this implementation corrects for decoherence due to loss of a photon in either arnie or bert. In this section, we describe an improvement of the mod 4 implementation. The improvement consists of measurement of individual photon-number-modulo-2 of each of arnie and bert, in addition to the joint-photon-number-modulo-
4 measurement. Thus, this improved mod 4 implementation is referred to as the \((\text{mod 4}) + P_a + P_b\) implementation, where \(P_a, P_b\) denote the individual photon-number-modulo-2 measurements of arnie, bert. As explained in the previous subsection, this improvement suppresses decoherence due to the loss of one photon in both arnie and bert and increases the success-rate of generating high fidelity entangled Bell states compared to the mod 4 implementation. Note that in absence of imperfections, the measurement of the individual parity of arnie and bert provides no additional information and advantage.

Incorporating this improvement in an experimental implementation poses no additional challenge compared to the mod 4 implementation as demonstrated in [79]. Further, the time required to make these additional measurements, with current circuit-QED parameters, is negligible compared to the typical qubit coherence times. This justifies neglecting the qubit decoherence for this part of the analysis. The theoretical calculations can be done in an analogous manner to that described in Sec. 5.4.2. The only difference comes in while computing the resultant state of Alice, Bob, arnie and bert after the individual photon-number-modulo-2 and the joint-photon-number-modulo-4 measurements, given by

\[
\rho_{\lambda}^{ABab} = \frac{\rho_{ABab}^\lambda}{\text{Tr}[\rho_{ABab}^\lambda]}.
\]

As before, the un-normalized density matrix \(\bar{\rho}_{ABab}^\lambda\) is obtained from \(\rho_{ABab}^\lambda\) [given in Eq. (5.24)] by restricting the summation of \(\gamma, \gamma', \delta, \delta'\) to be such that: \(\gamma + \delta = \lambda \mod 4, \gamma' + \delta' = \lambda \mod 4\). The only difference is that depending on the individual parity of arnie (bert) to be even or odd, the values of \(\gamma, \gamma'(\delta, \delta')\) are restricted to 0, 2 or 1, 3. The computation of the homodyne detection can also be done as before.

The results of the computation for the optimized total success-rate is plotted in Fig. 5.9. We show total success rate for the \((\text{mod 4}) + P_a + P_b\) implementation in the case of imperfect quantum efficiency, where we choose \(\eta_1 = \eta_2 = 0.8\). For comparison purposes, we show the left the same for the mod 4 implementation (also shown in Fig. 5.7, bottom right panel). While for \(\alpha \ll 1\), both implementations perform similarly, for larger values of \(\alpha\) when photon losses become more dominant, it is more advantageous to use the \((\text{mod 4}) + P_a + P_b\) implementation over the mod 4 implementation. This is because unlike the mod 4 implementation which corrects for loss of single photons to first order, the \((\text{mod 4}) + P_a + P_b\) implementation corrects for the loss of single photons in each of arnie and bert.

As before, for each value of cut-off fidelity and inefficiency, an optimal choice of \(\alpha\) can be obtained. This optimization is done numerically. The optimized success-rate as a function of inefficiency and cut-off fidelity is shown below (Fig. 5.10). We again choose \(\eta_1 = \eta_2\) for simplicity. For comparison purposes, we show the optimized probability of success for the mod 2 implementation (what is shown also in Fig. 5.8, left panel). The white curves in the right panel enclose the region where the \((\text{mod 4})\)
Figure 5.9: Total success rate for the mod 4 (left panel) and the (mod 4) +P_a + P_b (right panel) implementations are shown for the case of finite quantum efficiency, where we choose $\eta_1 = \eta_2 = 0.8$. While for $\alpha \ll 1$, both the mod 4 and the (mod 4) +P_a + P_b implementation perform similarly, for larger values of $\alpha$, the latter performs better than the other. This is because the mod 4 implementation corrects loss of single photon losses in either arnie or bert, while the (mod 4) +P_a + P_b implementation corrects for the loss of single photons in each of arnie and bert.

+P_a + P_b implementation has a higher success rate than the mod 2 implementation. In particular, for $\eta_1 = \eta_2 = 0.9$, only the (mod 4) +P_a + P_b implementation is able to give rise to Bell-states with overlap $\geq 0.95$ with a success-rate of $\sim 10^{-2}$, whereas the mod 2 implementation has a success rate less than $10^{-10}$ (the white rectangles in the plots). Even with efficiency values achievable in current circuit-QED systems of $\eta_1 = \eta_2 = 0.6$, with the (mod 4) +P_a + P_b implementation, one can generate entangled states with overlaps to Bell-states $\geq 0.8$ with a success-rate of $10^{-4}$ (the white circles in the plots). Note that, this protocol is not able to generate perfect fidelity Bell-states for efficiency parameters of around 0.6. This is because the mod 4 encoding protects against single photon losses to first order. For higher order protection, one will have to resort to different encodings [37,41,43,84].

5.5 Summary

To summarize, we have presented in this paper, a protocol to remotely entangle two distant, mutually non-interacting, stationary qubits. To that end, we have used a propagating ancilla qubit for each of the stationary qubit. In the first step, local entanglement is generated between each stationary qubit and its associated ancilla. Subsequently, a joint two-qubit QND measurement is performed on the propagating ancilla qubits, followed by individual single-qubit measurements on the same.
Figure 5.10: Optimized total success-probability $P^*$ for different cut-off fidelities and different choices of $\eta_1, \eta_2$ for the mod 2 (left panel) and the (mod 4) $+P_a + P_b$ (right panel) implementations. The optimization is done for the value of the parameter $\alpha$. We choose, for simplicity, $\eta_1 = \eta_2$. For both protocols, for an efficiency of $\eta_1 = \eta_2 = 0.8$, entangled states with overlaps to Bell-states $\sim 0.75$ are generated with near-unity success-rate. However, in contrast to the mod 2 implementation, the (mod 4) $+P_a + P_b$ implementation shows substantially more resilience to lower efficiency. The white curves in the right panel enclose the region where the (mod 4) $+P_a + P_b$ implementation has a higher success rate than the mod 2 implementation. For instance, the (mod 4) $+P_a + P_b$ implementation gives rise to Bell-states with fidelity $\geq 0.95$ with a success rate of $10^{-2}$ for an efficiency of $\eta_1 = \eta_2 = 0.9$ (the white rectangle in each plot). This should be compared with a success rate of less than $10^{-10}$ for the mod 2 implementation (left panel) and that of $10^{-4}$ for the mod 4 implementation (right panel of Fig. 5.8). Even with efficiency values achievable in current circuit-QED systems of $\eta_1 = \eta_2 = 0.6$, one can generate entangled states with overlaps to Bell-states $\geq 0.8$ with a success-rate of $10^{-4}$ (the white circle in each plot). However, for low enough efficiencies, the error correcting protocol ceases to be advantageous since higher order photon loss become more important. For higher order protection, one will have to resort to different encodings [37, 41, 43, 84].
Depending on the three measurement outcomes, the two stationary qubits are projected on to an entangled state. We have discussed two continuous variable implementation of our protocol. In the first implementation, the ancilla qubits were encoded in even and odd Schrödinger cat states. For this encoding, the two-qubit measurement was done by a joint-photon-number-modulo-2 measurement and the single-qubit measurements were performed by homodyne detections. Subsequently, we described a second implementation, where the ancilla qubits were encoded in mod 4 cat states. For this encoding, the two-qubit measurement was performed by a joint-photon-number-modulo-4 measurement and the single qubit measurements were performed by homodyne detections. We analyzed the resilience of the two implementations to finite quantum efficiency arising out of imperfections in realistic quantum systems. We described how with the mod 4 implementation, it is possible to suppress loss of coherence due to loss of a photon in either of the ancilla qubits. Lastly, we presented an improvement of the mod 4 implementation, where we made individual photon-number-modulo-2 measurements of the ancilla qubits, together with the joint-photon-number-modulo-4 measurement, by virtue of which we suppressed the decoherence due to loss of a photon in both the ancilla qubits. We demonstrated that it is indeed possible to trade-off a higher success rate, present in the mod 2 implementation, for a higher fidelity of the generated entangled state, present in the (mod 4) $+P_a + P_b$ implementation, using error correction.

Next, we point out two future directions of research that this work leads to. First, the use of homodyne detection as the single-qubit measurement in the final step of the mod 4 or the (mod 4) $+P_a + P_b$ implementation of our protocol lowers the success-rate of generating entangled states. This is because both arnie and bert are in superpositions of $|\alpha\rangle$ and $|\alpha\rangle$, and thus, irrespective of the choice of the quadrature, the homodyne measurement is always made on the complementary quadrature of the modes for one of the cat states. This gives rise to fringes in the resultant overlap to the Bell-states and lowers the success rate of generating the same. It will be worthwhile to explore alternatives for the homodyne detection to boost the success rate of the error correcting protocol. Second, the error correcting encoding we used is designed to protect against losses of single-photons to first order. This is why mod 4 implementation protects against loss of a photon in either of the ancilla qubits. By including individual parity measurements, in addition to the joint-photon-number-modulo-4 measurement, we corrected for the decoherence due to loss of a photon in both of the ancillas. However, with this encoding, higher order photon loss errors cannot be corrected. It will be interesting to explore different encodings of the ancilla qubits for protection against higher order photon losses [37, 41, 43, 84].
Chapter 6

Conclusion and perspectives

The primary goal of this thesis was to investigate concurrent remote entanglement protocols with continuous variables. Experimental realizations of these protocols test against local, non-contextual, hidden-variable descriptions of quantum mechanics, providing loophole-free tests of Bell’s inequalities. At the same time, they are crucial for large-scale quantum information processing.

We have presented two variations of a new protocol for concurrent remote entanglement to entangle two distant, mutually non-interacting, stationary qubits. To that end, we have used a propagating ancilla qubit for each of the stationary qubit. Following initialization, local entanglement was generated between each stationary qubit and its associated ancilla. In the next step, unlike existing schemes, which use a unitary operation at the signal processing stage to generate remote entanglement, we used a nonlinear, two-qubit, quantum non-demolition measurement ([XX] or [ZZ]) on the propagating ancilla qubits to erase the ‘which ancilla qubit is entangled to which stationary qubit information’. Lastly, linear single qubit measurements (Z-s or X-s) were performed on the ancillas. Depending on these three measurement outcomes, the two stationary qubits are projected on to a particular maximally entangled state. We proposed continuous variable implementations of the two variations of our protocol, where the ancilla qubits were encoded in superpositions of coherent states of propagating modes of light.

The work in this thesis paves the way to several avenues of research. First, the remote entanglement protocol using multiplication of quantum signals (see Chapter 3) used nonlinear scattering of nonclassical states of light and is fundamentally different from the existing protocols which use linear scattering for quantum signal processing. Superconducting circuit-QED systems have access to a strong, dispersive Josephson nonlinearity and provide a natural platform for implementing this
protocol. We proposed a Josephson circuit device (the Josephson Parametric Multiplier) which performs the nonlinear mode-mixing necessary for our protocol (see Chapter 4). Several other Josephson circuit devices were obtained in this thesis which performed nonlinear mode-mixing useful for quantum information processing with continuous variables (see Appendix D). These results indicate towards an unexplored plethora of driven-dissipative nonlinear interactions of bosons that could be engineered by driving specific configurations of Josephson junctions coupled to superconducting cavities. Not only could these be relevant for quantum information processing, but they are also indispensable for controlled investigation of non-equilibrium nonlinear interactions of bosonic modes. Further, pumping the strong and dispersive Josephson nonlinearity can potentially allow for exploration of strong, nonlinear effects in the quantum regime in coupled Josephson junction arrays. A particularly promising direction of research is investigation of quantum solitons in Josephson circuits, which could be useful for lossless transmission of quantum states between nodes of a quantum network.

Second, we proposed a remote entanglement protocol with quantum error correction to correct for decoherence due to photon loss to first order in realistic quantum devices (see Chapter 5). To that end, we encoded the logical qubits in superpositions of Schrödinger cat states of a given photon-number-parity. A natural extension of this work is to investigate quantum error-correcting remote entanglement protocols that correct photon losses to higher order. To that end, one could use other encodings proposed in [41, 43, 84]. This naturally raises the question of how to realize the logical operations for these encodings using superconducting circuit-QED systems. Pursuing this research direction not only will lead to remote entanglement protocols with error-correction for higher order photon losses than that addressed in this thesis, but also will result in a better understanding of quantum measurement theory for bosonic modes and how to best implement these measurements in physical systems.
Appendix A

Continuous generation and stabilization of Schrödinger cat states

A.1 Motivation

Non-classical input states such as single photons and superpositions of coherent states are the main candidates for universal quantum computation with linear optical circuits [12, 71, 72, 104]. This has stimulated experiments in which single-photon states are generated in a heralded [105–107] and on demand [108, 109] manner. Various experimental schemes have likewise produced and observed superposition of coherent states in optical systems in a heralded manner using photon subtraction [110–114]. In the context of cavity/circuit QED, such superposition states have been generated by mapping a qubit state to a coherent state superposition in a heralded manner [115] and on demand [116]. Here, we go a step further and we address the question of robustly stabilizing cavity photons in a superposition of coherent states. This could act as a continuous and deterministic source of non-classical input states in quantum information processing protocols.

To that end, we apply a dissipation engineering technique leading to an autonomous preparation and protection against decoherence of these states [117]. An earlier theoretical proposal within the framework of cavity QED with Rydberg atoms describe such a stabilization by an adequate engineered system-bath interaction [118]. The current proposal is adapted to photon states in quan-
tum superconducting circuits, and requires only the application of continuous-wave (CW) microwave drives of fixed frequencies and amplitudes, thus greatly simplifying an experimental implementation.

The first stage of our proposal builds on recent theoretical work in such systems [42] in which a bath was engineered such that photons are only exchanged in pairs. Such a nonlinear system-bath interaction was shown to stabilize the manifold spanned by two coherent states $|\alpha\rangle$ and $|\alpha\rangle$ (where $\alpha$, the coherent state amplitude parameter, is determined by a tunable external drive).

Very recently this proposal has been implemented successfully in an experimental set-up [119]. The dynamics generated by such an interaction conserves photon number parity: an initial vacuum state $|0\rangle$ would therefore converge to the even Schrödinger cat state $|C_\alpha^+\rangle = \sum_{n=0}^{\infty} c_{2n}|2n\rangle$, $c_m = \frac{e^{-|\alpha|^2/2}}{\sqrt{2(1+e^{-2|\alpha|^2})}} \frac{\alpha^m}{\sqrt{m!}}$. Similarly, an odd parity initial state will converge to the odd Schrödinger cat state $|C_\alpha^-\rangle = \sum_{n=0}^{\infty} c_{2n+1}|2n + 1\rangle$. Finally, an initial state with undefined parity will converge to a final state of undefined parity. In practice however, while one can add a two-photon bath interaction which transiently dominates the dynamics, there will always be a residual single-photon loss channel that will decohere these parity superpositions, leading to a statistical mixture of $|\alpha\rangle$ and $|\alpha\rangle$ in the steady-state.  

In this paper we present a theoretical proposal where we autonomously compensate for single photon loss and ensure the stabilization of a single superposition (e.g. $|C_\alpha^+\rangle = \frac{1}{\sqrt{2(1+e^{-2|\alpha|^2})}}(|\alpha\rangle + |\alpha\rangle$) in this manifold. Similarly to some recent autonomous stabilization protocols for superconducting qubits [37,121,122], we benefit from the high quality factors of the superconducting microwave resonators in presence of strong nonlinear interactions provided by Josephson elements. More precisely, we make use of dispersive (cross-Kerr) interaction between two cavity modes mediated by a transmon qubit coupled to both of them [123]. Working in the strong dispersive regime [124], we design an effective decay of the cavity mode from a cat state of odd parity to a cat state of even parity. This dissipation, together with the two-photon process, reduces the steady state from a manifold spanned by $\{ |C_\alpha^+\rangle, |C_\alpha^-\rangle \}$ to a unique state ($|C_\alpha^+\rangle$). The full system requires only a high Q “storage cavity”, coupled to two low-Q “readout cavities” through Josephson junctions and requires cavity decay and coupling parameters well within the reach of current technology. A trivial modification of the scheme leads to stabilization of $|C_\alpha^-\rangle$. Note that even though we use the term “readout” to refer to the dissipative baths, the information leaking through the ports associated with the two low-Q cavities does not need to be monitored. It suffices that it never returns to the stabilized “storage cavity”.

1. The changes in photon-number-parity resulting from single photon loss can, in principle, be continuously monitored [120] and compensated for, in a measurement based feedback scheme.
The paper is organized as follows: in Section A.2, we describe our dissipation engineering scheme that stabilizes an even Schrödinger cat state. In Section A.3, we describe the possible experimental implementation, engineering the Hamiltonian interactions and dissipation, that realizes the stabilization scheme. We sweep the parameters that are in principle tunable in an ongoing experiment to determine the optimal choice. Next, we perform adiabatic elimination of the faster dynamical variables to arrive at an effective interaction and dissipation for the storage cavity alone, providing analytic expressions for the various decay and interaction rates (Sec. A.3 and Appendix. A.4). We summarize our results in Sec. A.5.

### A.2 Two-photon process and parity selection

In this section, we briefly outline the interaction and dissipation scheme that gives rise to an even Schrödinger cat state ($|C^+_\alpha\rangle$) in the steady state regime. We assume, for the storage cavity, the existence of a single-photon decay channel which is the natural dominant decoherence channel in the absence of engineered system-bath interactions. We further assume that we have engineered two additional decay channels: the two-photon decay channel through which pairs of photons are lost into the environment (following previous work [42,125–128]), and a new, parity-selection decay channel, which leads to an effective transfer of population from the odd to the even photon number parity manifold. These decay channels are characterized by effective decay rates $\kappa_{2ph}$ and $\kappa_{ps}$, respectively, and we assume that we can engineer them to be much larger than the rate of single photon loss ($\kappa_{1ph}$) for the relevant cavity modes:

$$\kappa_{1ph} \ll \kappa_{2ph}, \kappa_{ps}.$$  \hspace{1cm} (A.1)

### A.2.1 Two-photon process

Consider a cavity mode coupled to a bath and a drive such that it absorbs or loses photons only in pairs. Denoting the annihilation operator for this two-photon driven-dissipative harmonic oscillator as $a_s$, the master equation for the mode is:

$$\frac{d\rho}{dt} = -i[H_{2ph}, \rho] + \kappa_{2ph} D(a_s^2)\rho + \kappa_{1ph} D(a_s)\rho,$$  \hspace{1cm} (A.2)

where $D(\hat{O})\rho = \hat{O}\rho\hat{O}^\dagger - \frac{1}{2}\hat{O}^\dagger\hat{O}\rho - \frac{1}{2}\rho\hat{O}\hat{O}^\dagger$ is the usual Lindblad operator, $H_{2ph} = i(\epsilon_{2ph} a_s^2 - \epsilon_{2ph}^* a_s^2)$ and $\epsilon_{2ph}$ is the two-photon drive strength. As noted, for $\kappa_{1ph} = 0$, one can show that starting from
vacuum \( \rho(t = 0) = |0\rangle \langle 0| \), the density matrix converges towards \( \rho(t \to \infty) = |C_\alpha^+ \rangle \langle C_\alpha^+| \), where 
\[
\alpha = \sqrt{2} \epsilon_{2\text{ph}} / \kappa_{2\text{ph}} \quad [42].
\]
In the presence of single photon loss, due to the random photon jumps, the cat state undergoes decoherence resulting in an incoherent mixture of \( |\alpha\rangle \) and \( |-\alpha\rangle \).

### A.2.2 Parity selection

In order to compensate for the decoherence due to single photon loss, we consider the action of effective jump operators of the form \( J_{2n} = |2n\rangle \langle 2n + 1| \), which acting on the odd number states bring it to the immediate lower even number state. This transfers the excitations from the odd parity manifold, which gets populated due to single photon loss, to the even parity manifold. Once the population is transferred to the even manifold, the two-photon process redistributes the population over the even manifold so as to reach the steady-state determined by the two-photon bath plus drive, \( |C_\alpha^+\rangle \). Let us consider, for simplicity, only one such operator: \( J_{2\tilde{n}} = |2\tilde{n}\rangle \langle 2\tilde{n} + 1| \), where \( 2\tilde{n} \) is the even integer closest to the average number of photons in the even cat \( |C_\alpha^+\rangle \). \(^2\) The two-photon process acts also on the odd manifold, where it redistributes population, with maximum around \( |2\tilde{n} + 1\rangle \), so as to funnel probability density towards the escape channel given by the jump operator, \( J_{2\tilde{n}} \). Thus, although by itself this jump operator only transfers the population from the Fock state \( |2\tilde{n} + 1\rangle \) to \( |2\tilde{n}\rangle \), together with the two-photon process, it drains the population from the odd to the even manifold (cf. Fig. A.1). The rate associated with this parity selection process will be denoted by \( \kappa_{\text{ps}} \). Thus, we can write down the master equation governing the stabilized evolution of the cavity mode:

\[
\frac{d\rho}{dt} = -i[H_{2\text{ph}}, \rho] + \kappa_{2\text{ph}} D(a_a^2)\rho + \kappa_{1\text{ph}} D(a_a)\rho + \kappa_{\text{ps}} D(J_{2\tilde{n}})\rho. \tag{A.3}
\]

In Fig. A.2, we show the results of simulation of this equation. On the right is shown the Wigner function for the final state for \( \alpha = 2 \) when all terms are present in the evolution equation. The interference fringes near the origin clearly show the negativity of the Wigner function. On the left we show the time evolution of the fidelity of the solution of the evolution equation with respect to the ideal target state for three cases. In the absence of single-photon loss \( (\kappa_{1\text{ph}} = 0) \), the fidelity approaches unity at a rate determined by \( \kappa_{2\text{ph}} \). When single-photon loss is added but not stabilization \( (\kappa_{1\text{ph}} \neq 0, \kappa_{\text{ps}} = 0) \) the fidelity grows initially but then decays to 0.5 as expected for the statistical mixture (asymptotic behavior data not shown here). When all three processes are

\(^2\) If the desired target state is \( |C_{\alpha}^-\rangle \), one needs to consider jump operators of the form \( J_{2\tilde{n} - 1} = |2\tilde{n} - 1\rangle \langle 2\tilde{n}| \).
present, the fidelity stabilizes at a value greater than 0.9. (For fidelity of a density matrix $\rho$ with respect to the target state $|C^+_\alpha\rangle$, we use the definition: $F = \langle C^+_\alpha|\rho|C^+_\alpha\rangle$). Here we choose two-photon dissipation rate and the parity selection rate to be $\kappa_{2\text{ph}} = 250\kappa_{1\text{ph}}, \kappa_{\text{ps}} = 760\kappa_{1\text{ph}}$, consistent with the required inequality (1) above.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{schematic.png}
\caption{Schematic for the stabilization of the “cat state”, $|C^+_{\alpha=2}\rangle$. The two-photon drive and dissipation (denoted by $H_{2\text{ph}}, \kappa_{2\text{ph}}D(a_s^2)$) act on the even and odd manifolds, shown in blue and orange respectively. In the absence of single photon loss and starting from vacuum, the odd manifold remains unpopulated, while the even manifold population is distributed to realize an even cat state. However, single photon loss (shown in red) denoted by $\kappa_{1\text{ph}}D(a_s)$ transfers some of the population to the odd manifold, where it is distributed as in an odd cat state due to the two-photon drive/dissipation. We propose to engineer a dissipation interaction from $|5\rangle$ to $|4\rangle$ (in green) denoted by $\kappa_{\text{ps}}D(J_4)$. This dissipation, together with the two-photon process, transfers excitations from the odd to even manifold and stabilizes the desired cat state.}
\end{figure}

The master equation we have studied is an idealized “cavity-only” system, whereas additional components will be required to realize the required baths and drives. In the following section we propose a possible experimental implementation of the aforementioned stabilization scheme. Subsequently we will analyze the reduction of this system to the an effective model described by the single cavity master equation.
Figure A.2: Fidelity with respect to the target state $|C^{-}_{\alpha=-2}\rangle$ (left panel) and Wigner function of the steady-state (right panel). The parity-selecting dissipation and the two-photon dissipation/drive, in the presence of single photon loss, stabilizes the even cat state. The dissipation rates are $\kappa_{2ph} = 250\kappa_{1ph}$, $\kappa_{ps} = 760\kappa_{1ph}$. The evolution of fidelity is shown in absence of single photon loss (blue dashed), presence of single photon loss and absence of parity selection (red double-dashed), and lastly, in presence of single photon loss and parity selection (green solid). Steady state Wigner function of stabilized cat state is shown in presence of single and two-photon loss and parity selection.

A.3 Proposed experimental implementation

We propose a three-cavity two-junction architecture, where a high-Q cavity (referred to as storage cavity $s$) is linked by small transmission lines to two low-Q cavities, referred to as readout cavities $r_1$ and $r_2$ as shown in Fig. A.3. Each transmission line has an in-line embedded Josephson junction, which by virtue of the Josephson nonlinearity provides a nonlinear coupling between the storage and readout cavities. The single photon loss rate of the storage cavity is given by $\kappa_{1ph}$, while that of the two readout cavities are given by $\kappa_{r_1}$ and $\kappa_{r_2}$ with the constraint:

$$\kappa_{1ph} \ll \kappa_{r_1}, \kappa_{r_2}. \quad (A.4)$$

The Hamiltonian of this device can be written as [129]:

$$H_{\text{cat-stab}} = \sum_k \hbar \omega_k a_k^\dagger a_k - E_{J_1} \left[ \cos \left( \frac{\Phi_1}{\phi_0} \right) + \frac{1}{2} \left( \frac{\Phi_1}{\phi_0} \right)^2 \right] - E_{J_2} \left[ \cos \left( \frac{\Phi_2}{\phi_0} \right) + \frac{1}{2} \left( \frac{\Phi_2}{\phi_0} \right)^2 \right]. \quad (A.5)$$

Here $E_{J_{1,2}}$ are the Josephson energy for the two junctions, $\omega_k$ are the bare frequencies of the modes $a_k$, $\phi_0 = \hbar/2e$ is the reduced flux quantum and $\Phi_{1,2}$ is the flux through the Josephson junction linking readout cavity $r_{1,2}$ to storage cavity $s$.

Here, only the fundamental modes of the three cavities are excited, annihilation operators
Figure A.3: Schematic of experimental set-up realizing the stabilization scheme. Josephson junction JJ \textsubscript{1} bridges the storage and readout cavity \textsubscript{r\scriptscriptstyle 1}. This, together with the stiff off-resonant pump at \( \omega_p = 2\omega_s - \omega_{r1} \), and the weak resonant drive at \( \omega_{r1} \) incident on \( r_1 \), gives rise to the two-photon drive and dissipation. Josephson junction JJ \textsubscript{2} bridges the storage and readout cavity \textsubscript{r\scriptscriptstyle 2} providing a nonlinear coupling between the modes \( a_s \) and low-Q mode \( a_{r2} \). An off-resonant pump incident on \( r_2 \) at frequency \( \omega_{p'} = (\omega_{r2} - \omega_s - 2\hbar\chi_{sr2})/2 \) gives rise to beam-splitter-like interaction between \( a_s \) and \( a_{r2} \): \( g_{ps}e^{\chi_{ps}t}a_s^\dagger a_{r2} + c.c. \). This beam-splitter-like interaction acts conditioned on the mode \( a_s \) having \( 2\hbar n + 1 \) photons in the storage cavity. When the condition is realized, this interaction transfers one quantum of excitation from the \( a_s \)-mode to the \( a_{r2} \)-mode, which is then lost irreversibly to the environment.

(frequencies) of which are denoted respectively by \( a_s(\omega_s), a_{r1}(\omega_{r1}) \) and \( a_{r2}(\omega_{r2}) \). The Josephson junctions ensure a nonlinear coupling of the modes \( a_s \) and \( a_{r1} \) and similarly between the modes \( a_s \) and \( a_{r2} \). This gives rise to self-Kerr and cross-Kerr interactions of the form: \( -\chi_{ff} f^\dagger f^2 \) and \( -\chi_{fg} (f^\dagger f)(g^\dagger g) \), where \( f, g \) correspond to the annihilation operators for the modes under consideration. Our stabilization scheme makes use of the following separation of time-scales (cf. Secs. A.3.1, A.3.2 for details):

\[
\chi_{sr1} \ll \kappa_{r1} \text{ and } \kappa_{r2} \ll \chi_{sr2}.
\]  

(A.6)

This separation of time-scales can be engineered by appropriately choosing the participation ratios of the modes interacting through the junction nonlinearity.
Figure A.4: Scattering processes taking place through the nonlinear elements. (a) One photon in readout mode $a_{r_1}$, together with one photon of pump at $\omega_p$ gets converted to two photons in mode $a_s$, giving rise to the two-photon drive. (b) Two photons of the mode $a_s$ are converted into one photon in pump mode at frequency $\omega_p$ and one photon in mode $a_{r_1}$, which then irreversibly decays to the environment, giving rise to two-photon dissipation. (c) One photon in the mode $a_s$, along with two photons in the pump with the adequate frequency $\omega_p'$, are converted conditionally into a photon in mode $a_{r_2}$, which then irreversibly decays to the environment. This process occurs only when the number of photons in the storage cavity is $2n + 1$, giving rise to the parity-selection mechanism.
A.3.1 Realizing two-photon process

We can engineer a non-linear interaction between the two modes $a_s$ and $a_{r_1}$ by means of a stiff (non-depleted), off-resonant pump incident on the readout cavity $r_1$. The frequency $\omega_p$ of the pump is chosen to be $\omega_p = 2\omega_s - \omega_{r_1}$. In addition, we drive the mode $a_{r_1}$ with a weak resonant tone of amplitude $\epsilon_{r_1}$ and frequency $\omega_{r_1}$. Following the same kind of analysis as in [129] and setting $\hbar = 1$ for the rest of this work, one can write the effective interaction Hamiltonian between the modes $a_s$ and $a_{r_1}$ as (see Fig. A.4):

$$H_{sr_1} = \omega_s a_s^\dagger a_s + \omega_{r_1} a_{r_1}^\dagger a_{r_1} + g_{2\text{ph}} (a_s^\dagger a_{r_1} + a_s a_{r_1}^\dagger) - \epsilon_{r_1} (a_{r_1} + a_{r_1}^\dagger)$$

$$- \frac{\chi_{ss}}{2} a_s^\dagger a_s^2 - \frac{\chi_{sr_1}}{2} a_{r_1}^\dagger a_{r_1}^2 - \chi_{sr_1} (a_s a_s^\dagger) (a_{r_1} a_{r_1}^\dagger),$$

where we have assumed the nonlinear coupling $g_{2\text{ph}}$ and drive amplitude $\epsilon_{r_1}$ to be real (phase of $g_{2\text{ph}}$ is fixed by the phase of the stiff pump at $\omega_p$) and neglected nonlinearity higher than fourth order in mode amplitudes. In writing Eq. (A.7), we have also included self-Kerr and cross-Kerr interaction terms of the modes $a_s, a_{r_1}$ arising out of $H_{\text{cat},\text{tab}}$. As shown in [42],

$$g_{2\text{ph}} = \frac{\epsilon_p}{\omega_p - \omega_{r_1}/2},$$

where $\epsilon_p$ is the amplitude of the pump drive. For the rate inequalities given by (Eq. (A.6)), the Hamiltonian (Eq. (A.7)), together with the decay of the low-Q mode $a_{r_1}$, gives rise to the two-photon drive and dissipation of Eq. (A.2) (cf. [119] and Chap. 12 of [77] for details of calculation).

A.3.2 Realizing parity selection

Next, we describe the interaction between the modes $a_s$ and $a_{r_2}$. We propose to engineer a beamsplitter-like interaction of the form $a_s a_{r_2}^\dagger + a_s^\dagger a_{r_2}$ conditioned on the number of photons in the $a_s$-mode being $2\tilde{n} + 1$. This interaction has the effect that when the mode $a_s$ has $2\tilde{n} + 1$ photons, a photon of the $a_s$ mode is destroyed, in turn creating a photon in the mode $a_{r_2}$, which is rapidly and irreversibly lost to the environment due to its low-Q nature of resonator $r_2$. This state-selective beam-splitter interaction is generated by a stiff pump incident on the readout cavity $r_2$ at frequency $\omega_{p'} = (\omega_{r_2} - \omega_s - 2\tilde{n}\chi_{sr_2})/2$ (see below for more details). To realize the number-selectivity of this interaction, we need to work in the strong dispersive regime of the storage cavity. This ensures that the beam-splitter interaction becomes off-resonant when the number of photons in mode $a_s$ is anything but $2\tilde{n} + 1$. The Hamiltonian describing the interaction between modes $a_s$ and $a_{r_2}$ is given
by (see Fig. A.4):
\[
\mathbf{H}_{sr2} = \omega_s a_s^\dagger a_s + \omega_{r1} a_{r1}^\dagger a_{r1} + g_{ps} \left( e^{2i\omega_p' t} a_s^\dagger a_{r2} + e^{-2i\omega_p' t} a_{r2}^\dagger a_s \right) - \frac{\chi_{sr2}}{2} a_s^2 a_s^2 - \chi_{sr2} (a_s^\dagger a_s)(a_{r2}^\dagger a_{r2}),
\]  
(A.9)

where \( g_{ps} \) is the strength of the beam-splitter interaction fixed by the pump amplitude \( (\epsilon_p') \) and is given by:
\[
g_{ps} = \sqrt{\chi_{sr2} \frac{\epsilon_{p'}}{\omega_{p'} - \omega_{r2}}}.
\]  
(A.10)

Due to the rate inequalities of Eq. (A.6), it is sufficient to keep only the cross-Kerr interaction \( -\chi_{sr2} (a_s^\dagger a_s)(a_{r2}^\dagger a_{r2}) \) for the calculation. The selectivity of the transition between the levels \(| 2\tilde{n} + 1 \rangle_{a_s} \otimes | 0 \rangle_{a_{r2}} \) and \(| 2\tilde{n} \rangle_{a_s} \otimes | 1 \rangle_{a_{r2}} \) is ensured by detuning the frequency of the stiff pump \( (\omega_{p'}) \) from \((\omega_{r2} - \omega_s)/2 \) by \( -\tilde{n} \chi_{sr2} \)
3. This leads to strong number selectivity when \( \chi_{sr2} \gg g_{ps} \). In addition, the cross-Kerr interaction has also to be stronger than the damping of the low-Q mode \( a_{r2} \), i.e. \( \chi_{sr2} \gg \kappa_{r2} \) so that the state-selectivity is not washed away by dissipation-induced level-broadening.

Moving to the rotating frame \( a_s \rightarrow a_s e^{-i\omega_s t}, a_{r1} \rightarrow a_{r1} e^{-i\omega_{r1} t}, a_{r2} \rightarrow a_{r2} e^{-i\omega_{r2} t + 2i\tilde{n} \chi_{sr2} t} \), we can now write down the master equation for the density matrix \( \rho_{sr1r2} \) for the full three-mode model associated with \( a_s, a_{r1} \) and \( a_{r2} \):
\[
\frac{d\rho_{sr1r2}}{dt} = -i [\tilde{\mathbf{H}}_{2ph} + \mathbf{H}_{ps} + \mathbf{H}_{\text{cross-Kerr}}, \rho_{sr1r2}] + [\kappa_{r1} D(a_{r1}) + \kappa_{r2} D(a_{r2}) + \kappa_{1ph} D(a_s)] \rho_{sr1r2},
\]  
(A.11)

where
\[
\tilde{\mathbf{H}}_{2ph} = g_{2ph} (a_s^\dagger a_{r1}^\dagger + a_s a_{r1}^\dagger) - \epsilon_{r1} (a_{r1}^\dagger a_{r1}),
\]
\[
\mathbf{H}_{ps} = g_{ps} (a_s a_{r2}^\dagger + a_s^\dagger a_{r2}),
\]
\[
\mathbf{H}_{\text{cross-Kerr}} = \chi_{sr2} (2\tilde{n} - a_s a_s^\dagger) a_{r2} a_{r2}^\dagger.
\]  
(A.12)

We now present the numerical results obtained from solving numerically the above three-mode master equation. In Fig. (A.5) we plot the fidelity with respect to the target cat state \(| C_{\alpha = 2}^+ \rangle \) upon variation of the parameters \( g_{2ph}/\kappa_{1ph} \) and \( g_{ps}/\kappa_{1ph} \). The choice of parameters is as follows:
\[
\kappa_{r1} = \kappa_{r2} = 10^3 \kappa_{1ph}, \chi_{sr2} = 2.5 \times 10^4 \kappa_{1ph}.
\]

The ratio \( \epsilon_{r1}/g_{2ph} = 4 \), so that the target cat state

3. In the case of stabilizing an odd cat state, \( \omega' \) is detuned from \((\omega_{r2} - \omega_s)/2 \) by \(-\tilde{n} + 1/2 \chi_{sr2} \), where \( 2\tilde{n} + 1 \) is the odd integer closest to the average number of photons in the target cat state.
Figure A.5: Fidelity with respect to the target cat state, obtained by solving Eq. (A.11), versus parameters $g_{2\text{ph}}/\kappa_{1\text{ph}}, g_{\text{ps}}/\kappa_{1\text{ph}}$. We choose $\kappa_1 = \kappa_2 = 10^3\kappa_{1\text{ph}}, \chi_{sr_1} = 2.5 \times 10^4\kappa_{1\text{ph}}$. The ratio of $\epsilon_1$ and $g_{2\text{ph}}$ is chosen to be 4 so that the target cat state is $|C^{+}_{\alpha=2}\rangle$. White square denotes the point of optimal fidelity, $\simeq 0.94$ for this choice of parameters ($g_{2\text{ph}} = 250\kappa_{1\text{ph}}, g_{\text{ps}} = 400\kappa_{1\text{ph}}$, cf. Fig. A.2)). The black square is the point in the shown range of parameters where the adiabatic elimination of Sec. A.4 works best.

is $|C^{+}_{\alpha=2}\rangle$. We see that for this choice of parameters, the optimal fidelity ($\sim 0.94$) is obtained for $g_{2\text{ph}} = 250\kappa_{1\text{ph}}, g_{\text{ps}} = 400\kappa_{1\text{ph}}$. The robustness of the scheme is indicated by the fact that for a large range of parameters, we find fidelities in excess of 90%. Note that $g_{2\text{ph}}$ cannot be increased arbitrarily; due to the inequalities (A.6), (A.8), $g_{2\text{ph}} \leq \kappa_1, g_{\text{ps}}$ also is bounded, by $\sqrt{\chi_{sr_2} \chi_{sr_1}}$ (cf. Eq. (A.10)), which is much larger than $\kappa_{r_2}$. For both these variables, these bounds are not reached in our simulations. Noting that the optimal fidelity of $\sim 94\%$ is mainly limited by single photon loss, a higher Q storage cavity, while the other parameters are fixed, would improve the target fidelity.

Another possibility to achieve high-fidelity cat states is to monitor and condition the generation, on the output of the readout mode $r_2$: by selecting the events where the output of the $r_2$ mode is in vacuum for a time duration of order $\kappa_{\text{ps}}^{-1}$, one can significantly increase the cat state fidelity. This selection excludes the events where a single-photon jump has happened but not yet been corrected.

Below, we show how the above three-mode master equation (Eq. (A.11)) can be reduced to the single-mode effective master equation (Eq. (A.3)), with the two-photon dissipation and parity-selection rates given by Eqs. (A.13), (A.14), where
\[ \kappa_{2\text{ph}} = \frac{4g_{2\text{ph}}^2}{\kappa_r}, \]  
\[ \kappa_{\text{ps}} = \frac{4\delta^2(2\tilde{n} + 1)}{1 + 4\delta^2(2\tilde{n} + 1)} \kappa_{r2}. \]  
(A.13)  
(A.14)

A.4 Elimination of fast dynamics

Due to the low-Q nature of the modes \( a_{r1} \) and \( a_{r2} \), we can eliminate their dynamics adiabatically to arrive at a reduced equation of motion for mode \( a_s \). Elimination of the \( a_{r1} \) mode can be done following Chap. 12 of [77]. This gives rise to a two-photon dissipation rate:

\[ \kappa_{2\text{ph}} = \frac{4g_{2\text{ph}}^2}{\kappa_r}. \]  
(A.15)

After eliminating the mode \( a_{r1} \), we proceed to eliminate the fast dynamics associated with the mode \( a_{r2} \). In the rotating frame of the Hamiltonian \( H_{\text{cross-Kerr}} \), the reduced master equation for the density matrix \( \rho_{sr2} \) for the modes \( a_s, a_{r2} \) is given by:

\[ \frac{d\rho_{sr2}}{dt} = -i\left[ i\left( \epsilon_{2\text{ph}} a_s a_s^\dagger - \epsilon_{2\text{ph}}^* a_s^2 \right) \Pi_{[0]a_{r2}} \rho_{sr2} \right] + \kappa_{2\text{ph}} D(a_s^2 \Pi_{[0]a_{r2}}) \rho_{sr2} + \mathcal{L}_{sr2} \rho_{sr2}, \]  
(A.16)

where

\[ \mathcal{L}_{sr2} \rho_{sr2} = -ig_{\text{ps}} \sum_{j=0}^\infty \left\{ \left[ \Pi_{[2\tilde{n}+1-j]a_s \otimes \langle j | a_{r2}} a_s a_{r2}^\dagger + a_s^\dagger a_{r2} a_s a_{r2}^\dagger \right] \rho_{sr2} \right\} + \kappa_{r2} D(a_{r2} \Pi_{[j]a_s}) \rho_{sr2} + \kappa_{1\text{ph}} D(a_s \Pi_{[j]a_{r2}}) \rho_{sr2} \]  
(A.17)

and \( \Pi_{[0]a_{r2}} = |0\rangle_{a_{r2}} a_{r2}^\dagger |0\rangle, \Pi_{[2\tilde{n}+1-j]a_s \otimes \langle j | a_{r2}} = |2\tilde{n} + 1 - j \rangle_{a_s} \otimes |j \rangle_{a_{r2}} a_s a_{r2} \rangle_{a_s} \langle j \rangle \otimes a_{r2} \langle 2\tilde{n} + 1 - j |.\) In writing Eqs. (A.16), (A.17), we have made use of the rotating wave approximation, assuming that \( \chi_{sr2} \gg \kappa_{r2}, g_{\text{ps}} \). In principle, for \( \kappa_{r2} > g_{\text{ps}} \), we can adiabatically eliminate the dynamics of the low-Q mode \( a_{r2} \). However, a direct calculation from Eqs. (A.16), (A.17) is difficult since any level of the mode \( a_{r2} \) can be excited. Instead, we approximately calculate an effective rate of transition of the system from the state \( |2\tilde{n}+1\rangle_{a_s} \otimes |0\rangle_{a_{r2}} \) to the state \( |2\tilde{n}\rangle_{a_s} \otimes |0\rangle_{a_{r2}} \) via the state \( |2\tilde{n}\rangle_{a_s} \otimes |1\rangle_{a_{r2}} \). Note that since the \( a_{r2} \)-mode is low-Q and there is no drive resonant at \( \omega_{r2} \), \( a_{r2} \) gets populated solely due to the interaction term of the form \( a_s a_{r2}^\dagger \) in Eq. (A.17). Hence we can expand the two-mode
density matrix $\rho_{sr2}$ as:

\[
\rho_{sr2} = \rho_{00}|0\rangle_{a_{r}a_{r}}\langle 0| + \delta (\rho_{01}|0\rangle_{a_{r}a_{r}}\langle 1| + \rho_{10}|1\rangle_{a_{r}a_{r}}\langle 0|) + \delta^2 (\rho_{11}|1\rangle_{a_{r}a_{r}}\langle 1| + \rho_{20}|2\rangle_{a_{r}a_{r}}\langle 0| + \rho_{02}|0\rangle_{a_{r}a_{r}}\langle 2|)
\]

\[+ \mathcal{O}(\delta^3), \quad (A.18)\]

where $\rho_{ij}, i,j = 0,1,2$ act on the Hilbert space of the $a_{s}$-mode. The natural small parameter of expansion is $\delta = g_{ph}/\kappa_{r2}$ (for similar analysis, cf. [130]). We will show that the short-lived states $\rho_{01}, \rho_{10}$ and $\rho_{11}$ can be adiabatically eliminated in favor of an effective dynamics of $\rho_{00}$. We will also see that $\rho_{20}, \rho_{02}$ can be dropped for a reduced dynamics in the sector of Hilbert space of $a_{s}$ which is of interest to us: span of $\{|2\tilde{n}\rangle_{a_{s}}, |2\tilde{n}+1\rangle_{a_{s}}\}$. For this calculation, we omit the two-photon drive/dissipation which acts only on $\rho_{00}$ and the single photon loss, the rate of which is much slower than the fast time-scale of the adiabatic elimination. These terms gives rise to a correction only in orders of $\mathcal{O}(\kappa_{1ph}/\kappa_{r2})$ and can be neglected. We will reinsert them at the end to get the final evolution of the reduced density matrix of mode $a_{s}$. Thus, from Eqs. (A.16),(A.17), we can write down an equation of motion for $\rho_{ij}, i,j = 0,1,2$ in dimensionless variable $\tau = \kappa_{r2} t$:

\[
\frac{d\rho_{00}}{d\tau} = -i\delta^2 (\Pi|2\tilde{n}+1\rangle_{a_{s}}a_{s}^\dagger \rho_{10} - \rho_{01}a_{s}\Pi|2\tilde{n}+1\rangle_{a_{s}}) + \delta^2 \sum_{n=0}^{\infty} \Pi|n\rangle_{a_{s}}\rho_{11}\Pi|n\rangle_{a_{s}},
\]

\[
\frac{d\rho_{11}}{d\tau} = -i (a_{s}\Pi|2\tilde{n}+1\rangle_{a_{s}}\rho_{01} - \rho_{10}\Pi|2\tilde{n}+1\rangle_{a_{s}}a_{s}^\dagger) - \rho_{11},
\]

\[
\frac{d\rho_{01}}{d\tau} = -i (\delta^2 \Pi|2\tilde{n}+1\rangle_{a_{s}}a_{s}^\dagger \rho_{11} - \rho_{00}\Pi|2\tilde{n}+1\rangle_{a_{s}}a_{s}^\dagger - \sqrt{2}\delta^2 \rho_{02}a_{s}\Pi|2\tilde{n}+2\rangle_{a_{s}}) - \frac{1}{2} \rho_{01},
\]

\[
\frac{d\rho_{10}}{d\tau} = -i (a_{s}\Pi|2\tilde{n}+1\rangle_{a_{s}}\rho_{00} + \sqrt{2}\delta^2 \Pi|2\tilde{n}+2\rangle_{a_{s}}a_{s}^\dagger \rho_{20} - \delta^2 \rho_{11}a_{s}\Pi|2\tilde{n}+1\rangle_{a_{s}}) - \frac{1}{2} \rho_{10},
\]

\[
\frac{d\rho_{20}}{d\tau} = -i \sqrt{2}\rho_{01}\Pi|2\tilde{n}+2\rangle_{a_{s}}a_{s}^\dagger - \rho_{20},
\]

\[
\frac{d\rho_{02}}{d\tau} = i \sqrt{2}\rho_{01}\Pi|2\tilde{n}+2\rangle_{a_{s}}a_{s}^\dagger - \rho_{02}. \quad (A.19)
\]

Define:

\[
\rho_{ij}^{m} = a_{s}\langle m|\rho_{ij}|m\rangle_{a_{s}}, \quad i,j = 0,1, m = 2\tilde{n}, 2\tilde{n}+1,
\]

\[
\tilde{\rho}_{ij} = a_{s}\langle 2\tilde{n}|\rho_{ij}|2\tilde{n}+1\rangle_{a_{s}}, \quad \tilde{\rho}_{ij} = a_{s}\langle 2\tilde{n}+1|\rho_{ij}|2\tilde{n}\rangle_{a_{s}}. \quad (A.20)
\]
Then, from Eqn. (A.19), we can write down:

\[
\begin{align*}
\frac{d\rho_{00}^{2n+1}}{d\tau} &= -i\delta^2\sqrt{2\bar{n} + 1}(\bar{\rho}_{10} - \bar{\rho}_{01}) + \delta^2\rho_{11}^{2n+1} \\
\frac{d\rho_{11}^{2n+1}}{d\tau} &= -\rho_{11}^{2n+1} \\
\frac{d\rho_{01}^{2n+1}}{d\tau} &= -i\delta^2\sqrt{2\bar{n} + 1}\bar{\rho}_{11} - \frac{1}{2}\rho_{01}^{2n+1} \\
\frac{d\rho_{10}^{2n+1}}{d\tau} &= i\delta^2\sqrt{2\bar{n} + 1}\bar{\rho}_{11} - \frac{1}{2}\rho_{10}^{2n+1}.
\end{align*}
\]  

(A.21)

We see that the dynamics of \(\rho_{11}^{2n+1}, \rho_{01}^{2n+1}\) and \(\rho_{10}^{2n+1}\) occur on a much faster time-scale than \(\rho_{00}^{2n+1}\) and thus, while performing adiabatic elimination, we can replace them by their steady-state values:

\[
\begin{align*}
[\rho_{11}^{2n+1}]_{s.s.} &= 0, [\rho_{01}^{2n+1}]_{s.s.} = -2i\delta^2\sqrt{2\bar{n} + 1}[\bar{\rho}_{11}]_{s.s.}, [\rho_{10}^{2n+1}]_{s.s.} = 2i\delta^2\sqrt{2\bar{n} + 1}[\bar{\rho}_{11}]_{s.s.}.
\end{align*}
\]  

(A.22)

Similarly, we can write down the equation of motion for \(\rho_{ij}^{2n}\):

\[
\begin{align*}
\frac{d\rho_{00}^{2n}}{d\tau} &= \delta^2\rho_{11}^{2n} \\
\frac{d\rho_{11}^{2n}}{d\tau} &= i\sqrt{2\bar{n} + 1}(\bar{\rho}_{10} - \bar{\rho}_{01}) - \rho_{11}^{2n} \\
\frac{d\rho_{01}^{2n}}{d\tau} &= i\sqrt{2\bar{n} + 1}\bar{\rho}_{00} - \frac{1}{2}\rho_{01}^{2n} \\
\frac{d\rho_{10}^{2n}}{d\tau} &= -i\sqrt{2\bar{n} + 1}\bar{\rho}_{00} - \frac{1}{2}\rho_{10}^{2n}.
\end{align*}
\]  

(A.23)

steady-state solutions of which give us:

\[
\begin{align*}
[\rho_{11}^{2n}]_{s.s.} &= -i\sqrt{2\bar{n} + 1}([\bar{\rho}_{01}]_{s.s.} - [\bar{\rho}_{10}]_{s.s.}); \\
[\rho_{01}^{2n}]_{s.s.} &= 2i\sqrt{2\bar{n} + 1}[\bar{\rho}_{00}]_{s.s.}; [\rho_{10}^{2n}]_{s.s.} = -2i\sqrt{2\bar{n} + 1}[\bar{\rho}_{00}]_{s.s.}.
\end{align*}
\]  

(A.24)

Using Eqns. (A.21), (A.22), (A.23), (A.24), we can write down equations of motion for \(\rho_{00}^{2n}\) and \(\rho_{00}^{2n+1}\):

\[
\begin{align*}
\frac{d\rho_{00}^{2n+1}}{d\tau} &= -i\delta^2\sqrt{2\bar{n} + 1}([\bar{\rho}_{10}]_{s.s.} - [\bar{\rho}_{01}]_{s.s.}), \\
\frac{d\rho_{00}^{2n}}{d\tau} &= i\delta^2\sqrt{2\bar{n} + 1}([\bar{\rho}_{10}]_{s.s.} - [\bar{\rho}_{01}]_{s.s.}).
\end{align*}
\]  

(A.25)

Note that \(\frac{d\rho_{00}^{2n+1}}{d\tau} + \frac{d\rho_{00}^{2n}}{d\tau} = 0\), which signifies that the population of the state \(|2\bar{n} + 1\rangle_a \otimes |0\rangle_{a_2}\) does
indeed decay to $|2\hat{n}\rangle_{a\downarrow} \otimes |0\rangle_{a\uparrow}$. To complete the analysis and get an explicit form of the rate of population transfer, we write down the equation of motion for $\bar{\rho}_{10}, \bar{\rho}_{01}$:

$$
\frac{d\bar{\rho}_{10}}{dt} = -i\sqrt{2\hat{n} + 1}(\rho_{00}^{2\hat{n}+1} - \delta^2 \rho_{11}^{2\hat{n}}) - \frac{1}{2}\bar{\rho}_{10},
$$

$$
\frac{d\bar{\rho}_{01}}{dt} = i\sqrt{2\hat{n} + 1}(\rho_{00}^{2\hat{n}+1} - \delta^2 \rho_{11}^{2\hat{n}}) - \frac{1}{2}\bar{\rho}_{01},
$$

steady state solutions of which are:

$$
[\bar{\rho}_{10}]_{s.s.} = -[\bar{\rho}_{01}]_{s.s.} = -2i\sqrt{2\hat{n} + 1}(\rho_{00}^{2\hat{n}+1} - \delta^2 \rho_{11}^{2\hat{n}}). \tag{A.26}
$$

Using Eqs. (A.24), (A.25), (A.26) and some tedious algebra, we have (in dimensional variables):

$$
\frac{d\rho_{00}^{2\hat{n}+1}}{dt} = -\kappa_{ps}\rho_{00}^{2\hat{n}+1}, \quad \frac{d\rho_{00}^{2\hat{n}}}{dt} = \kappa_{ps}\rho_{00}^{2\hat{n}+1}, \tag{A.27}
$$

where

$$
\kappa_{ps} = \frac{4\delta^2(2\hat{n} + 1)}{1 + 4\delta^2(2\hat{n} + 1)}\kappa_{r2}. \tag{A.28}
$$

Thus we have indeed derived an effective dynamics for the reduced density matrix of the storage mode: $\rho = \text{Tr}_{a\uparrow}[\rho_{sr2}]$ as given by Eq. (A.3) of Sec. A.2 with $\kappa_{ps}$ given by Eq. (A.28).

The key requirements for the above model reduction are the validity of the adiabatic approximation ($g_{2ph}/\kappa_{r1} \ll 1, g_{ps}/\kappa_{r2} \ll 1$) and the rotating wave approximation ($g_{ps}/\chi_{sr2} \ll 1, \kappa_{r2}/\chi_{sr2} \ll 1$).

In Fig. A.6, we compare the validity of the model-reduction for two choice of parameters (cf. Fig. A.5): the white square (purple curves) corresponding to $g_{ps} = 400\kappa_{1ph}, g_{2ph} = 250\kappa_{1ph}$ and $\epsilon_{r1} = 1000\kappa_{1ph}$ and the black square (orange curves) corresponding to $g_{ps} = 120\kappa_{1ph}, g_{2ph} = 50\kappa_{1ph}$ and $\epsilon_{r1} = 200\kappa_{1ph}$. For both sets of curves, $\chi_{sr2} = 2.5 \times 10^4\kappa_{1ph}, \kappa_{r2} = \kappa_{r1} = 1000\kappa_{1ph}$ and the target state is $|C_{a=2}\rangle$. The model-reduction (Eq. (A.3)) approaches the full three-mode master equation (Eq. (A.11)) as the adiabatic approximation ($g_{2ph}/\kappa_{r1} \ll 1, g_{ps}/\kappa_{r2} \ll 1$) and the rotating rotating wave approximation ($g_{ps}/\chi_{sr2} \ll 1, \kappa_{r2}/\chi_{sr2} \ll 1$) become more and more accurate.

### A.5 Summary

Following recent advances in the production of non-classical states of light, we have proposed a scheme to prepare, and protect against decoherence, Schrödinger cat states of given photon number parity. Relying only on the application of continuous-wave drives of fixed but carefully chosen...
Figure A.6: Comparison between the evolution of fidelities for the full three-mode master equation (Eq. (A.11)) in solid lines, and that obtained from the reduced dynamics (Eqs. (A.3), (A.28)) in dashed lines. The two sets of parameters are chosen from Fig. A.5: the white square (purple curves) corresponding to $g_{ps} = 400 \kappa_{1ph}, g_{2ph} = 250 \kappa_{1ph}$ and $\epsilon_{r_1} = 1000 \kappa_{1ph}$ and the black square (orange curves) corresponding to $g_{ps} = 120 \kappa_{1ph}, g_{2ph} = 50 \kappa_{1ph}$ and $\epsilon_{r_1} = 200 \kappa_{1ph}$. For both sets of curves, $\chi_{sr_2} = 2.5 \times 10^4 \kappa_{1ph}, \kappa_{r_2} = \kappa_{r_1} = 1000 \kappa_{1ph}$ and the target state is $|C_{\alpha=2}^+\rangle$. The model-reduction (Eq. (A.3)) approaches the full three-mode master equation (Eq. (A.11)) as the adiabatic approximation ($g_{2ph}/\kappa_{r_1} \ll 1, g_{ps}/\kappa_{r_2} \ll 1$) and the rotating rotating wave approximation ($g_{ps}/\chi_{sr_2} \ll 1, \kappa_{r_2}/\chi_{sr_2} \ll 1$) become more and more accurate.
frequencies, we are able to engineer an effective Hamiltonian and dissipation which stabilizes such states. The scheme is independent of the phase of the drives and appears to be robust with respect to the choice of their amplitudes. Numerical simulations illustrate that the required parameters are within reach of the ongoing experiments in the field of quantum superconducting circuits. Such a stabilized source of Schrödinger cat states is a valuable system component that could be integrated in existing quantum information processing schemes based only on linear optical scattering elements and amplifiers.
Appendix B

Quantum signal propagation on an infinite transmission line

B.1 Motivation

Quantum signals propagating along a transmission line are electromagnetic excitations of the transmission line, which involve only a few photons. The state of these excitations must display some degree of quantum purity for the signals to carry quantum information, which is the subject of interest in Josephson circuits. At the same time, a better understanding of these excitations is useful for understanding a quantum mechanical model of dissipation for these systems. In this section, we provide the basic mathematical background for the concept of photon applied to microwave electromagnetic excitations [76,131].

B.2 Hamiltonian description of a quantum transmission line

B.2.1 Derivation of the propagating modes of the transmission line from first principles

Consider an infinite transmission line, a one-dimensional electromagnetic medium characterized by a propagation velocity $v_p$ and a characteristic impedance $Z_c$. A microwave coaxial line serves as the canonical example of such medium (see Fig. B.1). Position along the line is indexed by the real number $x \in (-\infty, +\infty)$. We suppose that the line is ideal, with both $v_p$ and $Z_c$ independent of frequency $\omega$. The TEM modes propagating on this transmission line can be equivalently described by
Figure B.1: (a) Electromagnetic transmission line implemented as a coaxial cable. The parameter $x$ denotes the position along the line, $I$ denotes the average current along the line in the positive direction and $V$ the average voltage between the inner and outer conductors. The characteristic impedance and the propagation velocity are denoted by $Z_c$ and $v_p$, respectively. The line has a continuous density of modes in the limit where its length $2d \to \infty$. In (b), a ladder circuit model with cell dimension $\delta x$ models the infinite transmission line. Its capacitance and inductance per unit length are given by $L_{\ell} = L/\delta x$ and $C_{\ell} = C/\delta x$, respectively. In the limit where the signal frequency $\omega$ is small compared to $1/\sqrt{L C}$, $Z_c = \sqrt{L/C}$ and $v_p = 1/\sqrt{L_{\ell} C_{\ell}}$.

***

Following [24], we define a flux operator:

$$\phi(x, t) = \int_{-\infty}^{t} dt' V(x, t'),$$

where $V(x, t) = \partial_t \phi(x, t)$ is the local voltage operator at position $x$ on the transmission line at time $t$. The average voltage drop across a segment of length $\delta x$ with inductance $L_{\ell} \delta x$ is $-\delta x \partial_x \partial_t \langle \phi(x, t) \rangle$. The average flux through the inductance is given by $-\delta x \partial_x \langle \phi(x, t) \rangle$ and the operator for the current

$$v_p = \frac{1}{L_{\ell} C_{\ell}}, \quad Z_c = \sqrt{\frac{L_{\ell}}{C_{\ell}}}. \tag{B.1}$$

1. This model of a coaxial transmission line is due to Nyquist and corresponds to an actual microscopic description of the line, where $L_{\ell}, C_{\ell}$ correspond to the inductance and capacitance per unit length of the line. This should be differentiated from the Caldeira-Leggett model of resistance and impedance in terms of an infinite number of LC oscillators. The Caldeira-Leggett model describes the effective behavior of the resistor in terms of the LC oscillators even though underlying microscopic description of the resistor has nothing to do with LC oscillators.
flowing through the inductance is given by the usual relation:

\[ I(x, t) = -\frac{\partial x}{\partial t} \varphi(x, t)/L. \] (B.3)

The Lagrangian that describes the system is given by:

\[ L_{\text{line}} = \int_{-\infty}^{\infty} dx L_{\text{line}} = \int_{-\infty}^{\infty} dx \left\{ \frac{C_0}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2L} \left( \frac{\partial \varphi}{\partial x} \right)^2 \right\}, \] (B.4)

which through the Euler-Lagrange’s equation of motion results in the dispersionless wave propagation equation:

\[ \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{v_p^2} \frac{\partial^2 \varphi}{\partial t^2} = 0. \] (B.5)

The canonical conjugate momentum is the charge density \( \Pi(x, t) \):

\[ \Pi(x, t) = \frac{\delta L_{\text{line}}}{\delta \partial_t \varphi} = C_t \frac{\partial \varphi}{\partial t} = C_t V(x, t) \] (B.6)

and thus, the Hamiltonian describing the transmission is given as:

\[ H_{\text{line}} = \int_{-\infty}^{\infty} dx H_{\text{line}} = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2C_t} \Pi(x, t)^2 + \frac{1}{2L} \left( \frac{\partial \varphi}{\partial x} \right)^2 \right\}, \] (B.7)

We define the following Fourier transforms:

\[ \varphi(x, t) = \frac{1}{\sqrt{2\pi v_p}} \int_{-\infty}^{\infty} d\omega q(\omega, t) e^{i\omega x/v_p}, \] (B.8)

\[ \Pi(x, t) = \frac{1}{\sqrt{2\pi v_p}} \int_{-\infty}^{\infty} d\omega p(\omega, t) e^{i\omega x/v_p}. \] (B.9)

Here, the variable \( \omega/v_p \) denotes the wave-vector. The positive and negative values of \( \omega \) indicate wave-momentum in \( +x \) and \( -x \) direction. For a mode propagating with wave-vector \( \omega/v_p \), the energy is given by \( \hbar |\omega| \) (see below). Also, note that \( p(\omega, t) \), and \( not \ q(\omega, t) \), is the Fourier transform of the charge density. We adopt this notation because the flux and charge operators, \( \varphi(x, t) \) and \( \Pi(x, t) \), respectively correspond to the position and momentum operators of the corresponding mechanical oscillator. Since \( \varphi(x, t), \Pi(x, t) \) are Hermitian operators, it follows trivially that:

\[ q(\omega, t)^\dagger = q(-\omega, t), \quad p(\omega, t)^\dagger = p(-\omega, t). \] (B.10)
In terms of the Fourier transformed operators, the Hamiltonian is given by:

\[
H_{\text{line}} = \int_{-\infty}^{\infty} d\omega \left\{ \frac{1}{2C_\ell} p(\omega, t)p(-\omega, t) + \frac{\omega^2 C_\ell}{2} q(\omega, t)q(-\omega, t) \right\},
\]  

(B.11)

where we have used Eq. (B.1). Next, we define the quantity: \(a(\omega, t)\), which plays the role of annihilation operator for different modes of the transmission line:

\[
a(\omega, t) = \sqrt{\frac{|\omega|C_\ell}{2\hbar}} q(\omega, t) + \frac{i}{\sqrt{2\hbar|\omega|C_\ell}} p(\omega, t).
\]  

(B.12)

Note that with this definition, \(a(\omega, t)^\dagger \neq a(-\omega, t)\). Thus, the Hamiltonian can be rewritten in terms of these operators as follows:

\[
H_{\text{line}} = \int_{-\infty}^{\infty} d\omega \frac{\hbar|\omega|}{2} \left\{ a(\omega, t)a(\omega, t)^\dagger + a(\omega, t)^\dagger a(\omega, t) \right\}.
\]  

(B.13)

Next, we use the canonical quantization relation for the continuous field operators:

\[
[\varphi(x, t), \Pi(x', t)] = i\hbar \delta(x - x'),
\]  

(B.14)

which in the Fourier domain, becomes:

\[
[q(\omega, t), p(\omega', t)] = i\hbar \delta(\omega + \omega').
\]  

(B.15)

This leads to the following commutation relation for the annihilation operator \(a(\omega, t)\):

\[
[a(\omega, t), a(\omega', t)^\dagger] = \delta(\omega - \omega'), \quad [a(\omega, t), a(\omega', t)] = 0.
\]  

(B.16)

The Heisenberg equation of motion for the operator \(a(\omega, t)\) is given by:

\[
\frac{da(\omega, t)}{dt} = -i [a(\omega, t), H_{\text{line}}] = -i|\omega|a(\omega, t),
\]  

(B.17)

which can be solved to give:

\[
a(\omega, t) = e^{-i|\omega|(t-t_0)} a(\omega, t_0),
\]  

(B.18)

2. Although it might be tempting to identify \(a(-\omega, t)^\dagger = a(\omega, t)^\dagger\), we will refrain from doing so since the quantity \(a(\omega, t)^\dagger\) is devoid of any physical meaning.
where $t_0$ is some initial time. This, together with Eq. (B.8), leads to:

\[
q(\omega, t) = \sqrt{\frac{\hbar}{2|\omega|C_f}} \left\{ a(\omega, t_0)e^{-i|\omega|(t-t_0)} + a(-\omega, t_0)^\dagger e^{i|\omega|(t-t_0)} \right\} \tag{B.19}
\]

\[
p(\omega, t) = -i\sqrt{\frac{\hbar|\omega|C_f}{2}} \left\{ a(\omega, t_0)e^{-i|\omega|(t-t_0)} - a(-\omega, t_0)e^{i|\omega|(t-t_0)} \right\}. \tag{B.20}
\]

Now, we can solve for the field operators $\varphi(x, t), \Pi(x, t)$ arriving at:

\[
\varphi(x, t) = \sqrt{\frac{Z_c}{2\pi}} \int_{-\infty}^{\infty} d\omega \sqrt{\frac{\hbar}{2|\omega|}} e^{i\omega x/v_p} \left\{ a(\omega, t_0)e^{-i|\omega|(t-t_0)} + a(-\omega, t_0)^\dagger e^{i|\omega|(t-t_0)} \right\} \tag{B.21}
\]

\[
\Pi(x, t) = -\frac{i}{v_p\sqrt{2\pi Z_c}} \int_{-\infty}^{\infty} d\omega \sqrt{\frac{\hbar|\omega|}{2}} e^{i\omega x/v_p} \left\{ a(\omega, t_0)e^{-i|\omega|(t-t_0)} - a(-\omega, t_0)^\dagger e^{i|\omega|(t-t_0)} \right\} \tag{B.22}
\]

Note that, as expected, the operators $\varphi(x, t), \Pi(x, t)$ are Hermitian and have two traveling wave components corresponding to the two distinct traveling directions. From these quantities, it is easy to calculate the voltage and current operators using Eqs. (B.2), (B.3), leading to:

\[
V(x, t) = V^+(x, t) + V^-(x, t), \quad I(x, t) = I^+(x, t) + I^-(x, t), \tag{B.23}
\]

\[
I^+(x, t) = \frac{1}{Z_c} V^+(x, t), \quad I^-(x, t) = -\frac{1}{Z_c} V^-(x, t) \tag{B.24}
\]

where:

\[
V^\pm = -i\sqrt{\frac{Z_c}{2\pi}} \int_{0}^{\infty} d\omega \sqrt{\frac{\hbar\omega}{2}} \left\{ a(\pm\omega, t_0)e^{-i\omega(x+px)/v_p} e^{i\omega t_0} - h.c. \right\}. \tag{B.25}
\]

Here, we have expressed the current and voltage operators as superpositions of those operators propagating in opposite directions. As expected, the right(left)-propagating waves involve the field mode operators with positive (negative) wave-vectors. Next, we define the propagating wave amplitude.
operators in terms of these propagating current and voltage operators:

\[ A(\leftrightarrow)(x,t) = \frac{1}{2} \left( \frac{\mathbf{V}}{\sqrt{Z_c}}(x,t) \pm \sqrt{Z_c} \mathbf{I}(x,t) \right), \quad (B.26) \]

\[ A(\leftrightarrow)(x,t) = \frac{-i}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \sqrt{\frac{\hbar \omega}{2}} \left\{ a(\pm \omega, t_0) e^{-i\omega(\mp x/v_p)} e^{i\omega t_0} - h.c. \right\}, \quad (B.27) \]

Eq. (B.26) shows the spatial dependence can be obtained trivially from the \( A(\leftrightarrow)(x=0,t) \) by setting \( t \rightarrow t \mp x/v_p \). Thus,

\[ A(\leftrightarrow)(x=0,t) = \frac{-i}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \sqrt{\frac{\hbar \omega}{2}} \left\{ a(\pm \omega, t_0) e^{-i\omega(t-t_0)} - h.c. \right\}, \quad (B.28) \]

These propagating wave-amplitudes satisfy the following commutation relation:

\[ [A^{l_1}(x=0,t_1), A^{l_2}(x=0,t_2)] = \frac{i\hbar}{2} \frac{d}{dt} \delta(t_1 - t_2) \delta_{l_1 l_2}, \quad (B.29) \]

where \( l_1, l_2 = 0, 1 \) according as the denote the direction of propagation is \( \rightarrow, \leftarrow \). These traveling wave amplitudes are also obtained in the standard treatments of input-output theory describing a system with a few degrees of freedom connected to a semi-infinite transmission line [76, 85, 97, 99]. These propagating wave amplitude operators describe the power-flow in the transmission line. For instance, the net power flowing in the \( +x \) direction is given by:

\[ P = \langle A^\rightarrow (x,t) \rangle^2 - \langle A^\leftarrow (x,t) \rangle^2, \quad (B.30) \]

and is equivalent to the usual Poynting vector of electrodynamics. In terms of these propagating wave-amplitudes, the Hamiltonian can be written as:

\[ H_{\text{line}} = \frac{1}{v_p} \int_{-\infty}^{\infty} dx \left\{ A^\rightarrow (x,t)^2 + A^\leftarrow (x,t)^2 \right\}, \quad (B.31) \]

which expresses the fact that the total energy is the sum of the energies of these propagating waves.

Typically, in treatments of dispersionless quantum transmission lines (see Chap. 3 of [76], Chap. 3 of [97] and [132]), this is the point when the rotating wave approximation (RWA) is made before the discussion of traveling photon wavepackets. These treatments are sufficient when the spectral width of the traveling pulse is much smaller than the center frequency, which is usually the case in usual quantum optical systems operating with center frequencies in the THz range. However, for microwave circuit-QED systems operating with center frequencies in the GHz range, it is easy to
conceive of a temporal wave packet that is not well-described by these approximations. Therefore, we
go a step further and introduce the concept of a traveling-wave field-ladder operator without making
RWA. This is described below. We will look at these waves at a fixed point in space $x = 0$, since the
behavior at any other point is trivially deduced using the transformation: $t \rightarrow t - (+)x/v_p$ for
the right (left) propagating waves. For notational brevity, we will denote $A^\pm (x = 0, t)$ as $A^\pm (t)$.

Define the Fourier transforms of the propagating wave amplitudes as:

$$A^\leftrightarrow = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt A^\pm (t) e^{i\omega t}, \tag{B.32}$$

Here, we have used the squared parenthesis to separate the case when the Fourier transform is taken
with respect to time from the earlier case when the Fourier transform was with respect to space.
Since $A^\pm (t)$ in a Hermitian operator, it follows that $A^\leftrightarrow[\omega] = A^\leftrightarrow[-\omega]$. Using Eqs. (B.28), (B.32),
one readily obtains:

$$A^\leftrightarrow[\omega] = \begin{cases} -i \sqrt{\frac{\hbar}{2}} a(\pm \omega, t_0) e^{i\omega t_0}, \omega > 0, \\ i \sqrt{\frac{\hbar}{2}} a(\mp \omega, t_0)^\dagger e^{i\omega t_0}, \omega < 0. \end{cases} \tag{B.33}$$

These result in the following commutator relation:

$$[A^{l_1}_1[\omega_1], A^{l_2}_2[\omega_2]] = \text{sgn} (\omega_1 - \omega_2) \frac{\hbar}{4} \delta (\omega_1 + \omega_2) \delta l_1, l_2, \tag{B.34}$$

which is the frequency-domain counterpart of Eq. (B.29) and $l_1, l_2$ stand for the sense of propagation
as before.

Now, we are ready to define the traveling-wave field ladder operators:

$$a^l[\omega] = \frac{1}{\sqrt{\hbar |\omega|/2}} A^l[\omega]. \tag{B.35}$$

These traveling-wave field ladder operators $a^l[\omega]$ have commutation relations bearing a marked
resemblance to the ladder operators of a set of standing wave harmonic oscillators:

$$[a^{l_1}_1[\omega_1], a^{l_2}_2[\omega_2]] = \text{sgn} (\omega_1 - \omega_2) \delta (\omega_1 + \omega_2) \delta l_1, l_2. \tag{B.36}$$
This is somewhat clearer when we take into account that

\[ a^\dagger[\omega] = a^\dagger[-\omega]. \]  

(B.37)

In terms of the field operators \( a(\omega, t) \), these traveling-wave field ladder operators are given by:

\[ a \leftrightarrow [\omega] = -i a(\pm \omega, t_0) e^{i \omega t_0}, \quad \omega > 0, \]  

(B.38)

\[ = i a(\mp \omega, t_0)^\dagger e^{i \omega t_0}, \quad \omega < 0. \]  

(B.39)

As expected, we see that the right(left)-propagating waves involve the field operators \( a(\omega, t) \) with positive (negative) frequencies. Going back to the time domain, one can evaluate the propagating traveling-wave field ladder operators at \( x = 0 \) to be:

\[ a \leftrightarrow (t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega a \leftrightarrow [\omega] e^{-i \omega t} \]

\[ = -\frac{i}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega \{ a(\pm \omega, t_0) e^{-i \omega (t-t_0)} - a(\pm \omega, t_0)^\dagger e^{i \omega (t-t_0)} \}. \]  

(B.40)

It is important to note that \( a \leftrightarrow (t) \) is still a Hermitian operator and satisfies the commutation relation:

\[ [a^{l_1}(t_1), a^{l_2}(t_2)] = \frac{i}{\pi} p.p \frac{1}{t_1 - t_2} \delta_{l_1, l_2}. \]  

(B.41)

**B.2.2 Definition of the state of a traveling photon without the rotating wave approximation**

In order to properly define the photons of the line, one needs to introduce an orthonormal signal basis consisting of “first-quantization” wavelets \( w_{mp}^l(t) \) such that

\[ \int_{-\infty}^{t_1} dt \ w_{m_1 p_1}^l(t) w_{m_2 p_2}^l(t)^* = \delta_{m_1,m_2} \delta_{p_1,p_2} \delta_{l_1,l_2}, \]  

(B.42)

\[ w_{mp}^l(t)^* = w_{-mp}^l(t), \]  

(B.43)

\[ \sum_{m=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} w_{mp}^l(t_1) w_{-mp}^l(t_2) = \delta (t_1 - t_2). \]  

(B.44)

The pair of indices \( (|m|, p) \in \mathbb{N}^+ \times \mathbb{Z} \) defines a propagating temporal mode of the line, and the combined amplitudes of the two corresponding wavelets can be seen as an elementary degree of freedom of the field. There are two conjugate wavelets per mode since the phase space of each mode
is bi-dimensional.

It is necessary that the support of \( w_{mp}[\omega] \), the Fourier transform of \( w_{mp}(t) \), is entirely contained in the positive frequency sector if \( m > 0 \) and in the negative frequency sector if \( m < 0 \).

\[
w_{mp}[\omega] = w_{mp}[\omega] \Theta(\omega) \quad \text{if} \quad m > 0, \quad (B.45)
\]

\[
w_{mp}[\omega] = w_{mp}[\omega] \Theta(-\omega) \quad \text{if} \quad m < 0. \quad (B.46)
\]

In these last expressions, \( \Theta(\omega) \) is the Heaviside function.\(^3\)

This complete wavelet basis is a purely classical signal processing concept and its existence solely results from the property of the signals to be square-integrable functions. Any continuous signal \( f(t) \) such that \( \int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty \) can indeed be decomposed into a countable infinite number of elementary signals

\[
f(t) = \sum_{m=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} f_{-mp} w_{mp}(t), \quad (B.47)
\]

\[
f_{mp} = \int_{-\infty}^{+\infty} dt \ w_{mp}(t) f(t). \quad (B.48)
\]

A common example of such a wavelet is the Shannon wavelet

\[
w_{mp}(t) = 2^{\tau/2} \pi \frac{\sin \left( \frac{\pi}{\tau}(t - pr) \right)}{t} e^{i2\pi mt/\tau} \quad (B.49)
\]

whose Fourier transform is

\[
w_{mp}[\omega] = \sqrt{\frac{\tau}{2\pi}} \frac{1_{x_1,x_2}(m-1/2), \frac{\pi}{\tau}(m+1/2)}{\frac{\pi}{\tau}(m)} (\omega) e^{i\omega \tau}, \quad (B.50)
\]

where \( 1_{x_1,x_2}(x) \) is the indicator function which is 0 everywhere except in the interval \([x_1, x_2]\), where its value is unity. Many other useful bases, involving more continuous wavelets, exist [133]. In the above example, the center frequency and time location of the wavelet is \( 2\pi m/\tau \) and \( pt \), respectively (in order to form a complete basis, the pitch in frequency \( \Delta \omega \) and pitch in time \( \Delta t \) of the wavelet basis has to satisfy \( \Delta \omega, \Delta t \leq 2\pi \)).

The discreteness of the signal component indices is the justification for the term “first-quantization” and no quantum mechanics is involved here since all functions are at this stage c-number valued. Second-quantization intervenes when we define the discrete ladder field operators, with indices \( m > 0 \)

\(^3\) The index value \( m = 0 \) corresponds to special wavelets that have to be treated separately.
and $p$

\[
\psi_{mp}^l = \int_{-\infty}^{+\infty} d\omega w_{mp}^l(\omega) a^l(\omega), \quad (B.51)
\]

\[
\psi_{-mp}^l = \psi_{mp}^{l\dagger}, \quad (B.52)
\]

We introduce the short-hand $\mu = (l, |m|, p)$ as the index of the spatio-temporal mode, also called the flying oscillator. The photon-number operator is given by:

\[
n_\mu = \psi_\mu^{\dagger} \psi_\mu \quad (B.53)
\]

and the discrete ladder operators $\psi_\mu$ satisfy the same commutation relation as standing mode ladder operators:

\[
[\psi_{\mu_1}, \psi_{\mu_2}^{\dagger}] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\omega_1 d\omega_2 w_{m_1 p_1}^{l_1}(\omega_1) w_{m_2 p_2}^{l_2}(\omega_2)^* [a^{l_1}(\omega_1), a^{l_2}(\omega_2)] = \delta_{\mu_1, \mu_2}. \quad (B.54)
\]

An important remark can be made: if the photon amplitude operator $\psi_\mu$ is non-hermitian, this is only because its first quantization component $w_{mp}^l(t)$ is complex. Its second quantization component $a^l(t)$ is an hermitian operator. It is also important to note that, in general, the frequency of a photon is ill-defined, in contrast with what could be inferred from elementary introductions to quantum mechanics. This feature happens as soon as the duration of the wavelet corresponding to that particular photon is not very long compared with the inverse of the wavelet center frequency. Thus the concept of photon for a propagating signal has to be clearly distinguished from an energy quantum. A propagating photon is an elementary excitation of the field carrying a quantum of action, and corresponds to a field wavefunction orthogonal to the vacuum.

\[
|\Psi_{1\mu}\rangle = \psi_\mu^{\dagger} |\text{vac}\rangle, \quad (B.55)
\]

\[
\langle \text{vac} | \Psi_{1\mu}\rangle = 0. \quad (B.56)
\]

A wavelet can contain several photons in mode $\mu$

\[
|\Psi_{n\mu}\rangle = \frac{1}{\sqrt{n!}} (\psi_\mu^{\dagger})^n |\text{vac}\rangle \quad (B.57)
\]

and each multi-photon state (Fock state) is orthogonal to the others
\[ \langle \Psi_{n_2 \mu} | \Psi_{n_1 \mu} \rangle = \delta_{n_1 n_2}. \] (B.58)

Several modes can simultaneously be excited

\[ | \Psi_{n_1, \mu_1; n_2, \mu_2; n_3, \mu_3; \ldots} \rangle = \frac{1}{\sqrt{n_1!}} (\psi^\dagger_{\mu_1})^{n_1} \frac{1}{\sqrt{n_2!}} (\psi^\dagger_{\mu_2})^{n_2} \frac{1}{\sqrt{n_3!}} (\psi^\dagger_{\mu_3})^{n_3} \ldots | \text{vac} \rangle. \] (B.59)

The sequence of indices \( \sigma = (n_1, \mu_1; n_2, \mu_2; n_3, \mu_3; \ldots) \) is a mode photon occupancy configuration. Finally, the most general wavefunction of the field of the transmission line(s) is a superposition of all field photon configurations in all the spatio-temporal modes of the line(s):

\[ | \Psi \rangle = \sum_{\sigma} C_{\sigma} | \Psi_{\sigma} \rangle. \] (B.60)

There are exponentially many more quantum coefficients \( C_{\sigma} \) than the classical coefficients \( f_{\mu} \) in Eq. (B.48)! And it is also important to understand that a state with a well defined number of photons in a certain wavelet basis can be fully entangled in another basis.

A wavelet can also support a so-called coherent state instead of a well defined number of photons:

\[ | \alpha_{\mu} \rangle = e^{-|\alpha_{\mu}|^2/2} \sum_{n} \frac{\alpha_{\mu}^n}{\sqrt{n!}} | \Psi_{n \mu} \rangle, \] (B.61)

\[ = e^{-|\alpha_{\mu}|^2/2} e^{\alpha_{\mu} \psi^\dagger_{\mu} | \text{vac} \rangle}, \] (B.62)

and if all wavelets are in a coherent state, we obtain a coherent field state

\[ | \Psi \{ \alpha \} \rangle = \prod_{\mu} | \alpha_{\mu} \rangle \] (B.63)

\[ = e^{-\sum_{\mu} (|\alpha_{\mu}|^2/2 - \alpha_{\mu} \psi^\dagger_{\mu} )} | \text{vac} \rangle. \] (B.64)

Thus, the set of complex coefficients \( \alpha_{\mu} \) plays the role of the coefficients \( f_{\mu} \) in Eq. (B.48). Somewhat surprisingly, this property of being a coherent state remains true in every wavelet basis (as can be inferred from the quadratic form in the exponent of Eq. (B.64)).
B.2.3 Compatibility with the rotating wave approximation

In this section, we make RWA in our treatment of the quantized transmission line and demonstrate that our description is compatible with existing RWA treatments of the same. To that end, consider the traveling-mode field ladder operators obtained in Eq. (B.40). Now, we restrict ourselves to a narrow band of frequencies around a central frequency: \( \Omega \). For brevity, we will show the compatibility with the right-propagating waves. Defining

\[
\omega' \equiv \Omega - \omega,
\]

\( a \rightarrow (t) \) can be re-written as:

\[
a \rightarrow (t) = -i \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\infty} d\omega' \{ a(\omega' + \Omega, t_0) e^{i\Omega t_0} e^{i(\Omega + \omega') t} - a(\omega' + \Omega, t_0) \dagger e^{-i\Omega t_0} e^{-i(\Omega + \omega') t} \}. \tag{B.65}
\]

Defining new field-operators:

\[
b(\omega', t_0) \equiv -i a(\omega' + \Omega, t_0) e^{i\Omega t_0}, \tag{B.66}
\]

we get:

\[
a \rightarrow (t) = e^{-i\Omega t} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\infty} d\omega' b(\omega', t_0) e^{-i\omega'(t-t_0)} \right\} + e^{i\Omega t} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\infty} d\omega' b(\omega', t_0) \dagger e^{i\omega'(t-t_0)} \right\}. \tag{B.67}
\]

Now, consider the case when we are only interested in \( \omega' \) that lies in the vicinity of \( \Omega \). In that case, it is safe to drop the counter-rotating term rotating at \( -\Omega \) (the second term in the right hand side of Eq. (B.67)). This is the rotating wave approximation, resulting in:

\[
a_{\text{RWA}} \rightarrow (t) = e^{-i\Omega t} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\infty} d\omega' b(\omega', t_0) e^{-i\omega'(t-t_0)} \right\}. \tag{B.68}
\]

In standard treatments [76, 97], the integral is now extended to \(-\infty\), leading to the following commutation relation for the \( a_{\text{RWA}} \rightarrow (t) \):

\[
[a_{\text{RWA}} \rightarrow (t_1), a_{\text{RWA}} \rightarrow (t_2) \dagger] = \delta(t_1 - t_2). \tag{B.69}
\]

This can be easily checked by noting that the operators \( b(\omega', t_0) \) obey the same commutation relations as \( a(\omega', t_0) \). Similar set of analysis can be done for the left-propagating waves. While it is mathematically convenient to set \( \Omega \rightarrow \infty \) to express the commutation relation of the traveling field-operators under RWA in a compact form, one should be careful in interpreting this equation.

From Eq. (B.66), it is clear that one is never truly considering negative frequency values. The smallest frequencies considered are merely those positive frequencies which are far lower than \( \Omega \). The operator \( a_{\text{RWA}} \rightarrow (t) \) is what is present in the right-hand side of the Quantum Langevin Equation.
B.3 Summary

To summarize, we have derived the propagating mode field operators for an infinite transmission line in 1D. Furthermore, we have defined the concept of a traveling photon wavepacket without invoking rotating wave approximation. Our results are consistent with the existing treatment of the same under RWA. This formalism is essential for investigation of soliton wave propagation in engineered nonlinear transmission lines, which could be useful for quantum information processing (in particular, quantum state transfer between two distant nodes in a quantum network) in these systems.
Appendix C

Wavelet basis for the harmonic oscillator

C.1 Motivation

Consider a quantum harmonic oscillator. The Hamiltonian for this system is given by:

\[ H = \hbar \omega (a^\dagger a + \frac{1}{2}) , \]  

where \( \omega \) is the frequency of the harmonic oscillator and \( a \) is the annihilation operator of the relevant mode of the same. Many basis sets have been proposed that span the state-space of this system [131].

To that end, one can consider Fock states, denoted by \( |n\rangle \), which are defined to be the eigenstates of the number operator \( a^\dagger a \):

\[ a^\dagger a |n\rangle = n |n\rangle . \]  

Application of the annihilation and creation operators on these states are given by:

\[ a |n\rangle = \sqrt{n} |n-1\rangle , \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \]  

and the vacuum state is annihilated by the operator \( a : a |0\rangle = 0 \). The Fock states can be obtained by successive application of the creation operator from the vacuum:

\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle , \quad n = 0, 1, 2, \ldots \]
These states are orthonormal and complete:

$$\langle n|m \rangle = \delta_{n,m}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = 1.$$  \hspace{1cm} (C.5)

Since the Fock states are eigenstates of the number operator, they have infinite phase uncertainty, which makes descriptions of states with well-defined phase with Fock states more cumbersome. This is somewhat remedied by the coherent states. These states, denoted by $|\alpha\rangle$, are eigenstates of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle$$  \hspace{1cm} (C.6)

and $|\alpha\rangle$ is defined in terms of Fock states below below:

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$  \hspace{1cm} (C.7)

The coherent states can also be defined as displaced vacuum states:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.$$  \hspace{1cm} (C.8)

These states are quasi-orthogonal and over-complete:

$$\langle \beta|\alpha \rangle = \exp\left[-\frac{1}{2} (|\alpha|^2 + |\beta|^2) + \alpha \beta^* \right], \quad \int \frac{d^2 \alpha}{\pi} |\alpha\rangle \langle \alpha| = 1.$$  \hspace{1cm} (C.9)

While the coherent states have a well-defined phase and are (over-)complete, their quasi-orthogonality makes them less than ideal as a choice of basis states\(^1\). In what follows, we will define a new set of basis states for the harmonic oscillator, which, on one hand, are orthogonal and complete, and on the other hand, also have finite phase uncertainty.

\(^1\) Finitely squeezed states are also possible candidates for a basis set, but they suffer from similar drawbacks as coherent states [131]. In the limit of infinite squeezing, these states become equivalent to the position or momentum eigenstates, which are orthonormal (in a continuous sense) and complete.
C.2 Definition and properties

Consider the following superposition of Fock states:

\[
|r, t\rangle = \frac{1}{\sqrt{2r}} \sum_{n=2r-1}^{4r-2} e^{\frac{i\pi n}{r}} |n\rangle, \quad t = 0, 1, 2, \ldots, 2r - 1, r = 1, 2, \ldots,
\]

\[
= |0\rangle, \quad r = 0. \tag{C.10}
\]

Proof that these states are orthonormal:

\[
\langle r, t | r', t' \rangle = \frac{\delta_{r,r'}}{2r} \sum_{n=2r-1}^{4r-2} e^{\frac{i\pi n}{r} (t' - t)}
\]

\[
= \frac{\delta_{r,r'}}{2r} e^{\frac{i\pi n (r'-r)}{r}} \frac{1 - e^{i2\pi (t' - t)}}{1 - e^{\frac{i\pi n (r'-r)}{r}}}
\]

\[
= \delta_{r,r'} \delta_{t,t'}. \tag{C.11}
\]

Fig. C.2 shows the Wigner distributions (normalized to $\pm 1$) for $|0, 0\rangle$, $|1, 0\rangle$, $|1, 1\rangle^2$.

Average number of photons in these states are given by:

\[
\langle r, t | a^\dagger a | r, t \rangle = \frac{3}{2} (2r - 1), r \neq 0
\]

\[
= 0, r = 0 \tag{C.12}
\]

and the average photon-number uncertainty for these states is:

\[
\langle r, t | (\Delta n)^2 | r, t \rangle = \langle r, t | n^2 | r, t \rangle - \langle r, t | n | r, t \rangle^2 = \frac{1}{12} (4r^2 - 1), r \neq 0,
\]

\[
= 0, r = 0. \tag{C.13}
\]

C.3 Summary

To summarize, we have presented in this chapter, a new basis set for the state-space of a quantum harmonic oscillator. The elements of this set are orthonormal and complete and have finite phase uncertainty, making them valuable for description of non-classical states of light.

---

2. For any density matrix, the Wigner distribution is defined as: $W(\alpha, \alpha^*) = \frac{2}{\pi} \text{Tr} [D(-\alpha) \rho D(\alpha) \mathcal{P}]$, where $\mathcal{P} = e^{i\pi a^\dagger a}$ is the photon-number-parity operator. Here, we plot $\frac{\pi}{2} W(\alpha, \alpha^*)$. 

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Figure C.1: Wigner distributions, scaled to range from $-1$ to $1$, are plotted for $|0, 0\rangle$, $|1, 0\rangle$, $|1, 1\rangle$. 

Figure C.1: Wigner distributions, scaled to range from $-1$ to $1$, are plotted for $|0, 0\rangle$, $|1, 0\rangle$, $|1, 1\rangle$. 

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Figure C.2: Wigner distributions, scaled to range from $-1$ to $1$, are plotted for wavelets $|2, 0\rangle$, $|2, 1\rangle$, $|2, 2\rangle$, $|2, 3\rangle$. 
Appendix D

Nonlinear mode-mixing in Josephson circuits

D.1 Motivation

An essential component of quantum computation with continuous variables (e.g., modes of electromagnetic field) are non-classical states i.e., states with negativity in their Wigner functions [26–30]. These states can be generated by engineering a Hamiltonian with terms higher than quadratic in mode-amplitude, for instance the Kerr Hamiltonian, which is quartic [87]. The aforementioned Hamiltonian, together with linear scattering elements like beam-splitters, drives and squeezers, is sufficient to perform arbitrary polynomial transformations of the mode variables [25]. Such a non-linear Hamiltonian can be generated in superconducting circuit-QED systems using a robust, simple and non-dissipative circuit element: the Josephson tunnel junction. When operated below the critical current ($I_0$), a Josephson junction behaves as a pure nonlinear inductor, with an inductance $L_J = \varphi_0/(I_0 \cos \delta)$, where $\delta$ is the gauge-invariant phase threading the junction and $\varphi_0 = \hbar/(2e)$ is the reduced flux quantum. This gives rise to strong, tunable, non-dissipative non-linear coupling of the electromagnetic modes and the strength of this nonlinearity can be much larger than the linear coupling and dissipation rates of the circuit at the single photon level.

By coupling together several Josephson junctions, one can give rise to a system whose normal modes participate in a strong nonlinear interaction. Parametric amplifiers and frequency convertors based on Josephson junction circuits have been proposed and demonstrated recently [35, 36, 88–96]. These devices are described by a nonlinear Hamiltonian which is cubic or quadratic, which
is subsequently driven with a stiff, off-resonant pump. This gives rise to an effective quadratic parametric amplifier/converter Hamiltonian.

Recently, two Josephson circuit devices have been proposed which give rise to nonlinear mode-mixing with tunable, higher than quadratic Hamiltonians. First, a nonlinear four-wave mixing device with four distinct modes, the Josephson Parametric Multiplier (JPM) has been proposed [101]. This device gives rise to a tunable, three-wave mixing Hamiltonian in presence of a stiff, off-resonant pump, which is useful for remote entanglement of two distant qubits. Second, a nonlinear six-wave mixing device with three distinct modes has been proposed [42]. This device gives rise to a tunable, five-wave mixing Hamiltonian necessary for generating superpositions of Schrödinger cat states of a given photon-number-parity in an autonomous manner. These states can be used to encode logical states of a qubit and protected from decoherence by performing quantum error correction [134].

In this chapter, we describe two other Josephson junction circuits that give rise to higher than quadratic Hamiltonians useful for quantum information processing. For simplicity, we will only describe cases where all the participating junctions are identical and sufficiently large so that their charging energies can be neglected. The chapter is organized as follows. First, we describe a device that gives rise to a tunable, three-wave mixing Hamiltonian with one or two distinct mode(s) in Sec. D.2. Then, we describe the Josephson Cube Modulator (JCM) which gives rise to three, tunable, three-wave mixing Hamiltonians in Sec. D.3.

### D.2 Tunable three-wave mixing with one and two modes

Consider three Josephson junctions, nominally identical, arranged in the form of ‘Y’ as shown in Fig. D.1. Define node fluxes at nodes $x, y, z, w$ as:

$$\Phi_i = \int_{-\infty}^{t} V_i(t) \, dt, \quad i = x, y, z, w$$  \hspace{1cm} (D.1)
Figure D.1: This Josephson circuit device has three nominally identical Josephson junctions and has three mutually orthogonal, interacting normal (electrical) modes, shown in (a), (b) and (c). A fourth mode remains decoupled from the rest and is not shown here for brevity. The Josephson nonlinearity, together with the off-resonant pump, gives rise to a nonlinear interaction.

where \( V_i \) is potential at the node \( i \). There are four (normalized) normal modes of this device, denoted by \( \Phi_a, \Phi_b, \Phi_c, \Phi_d \), given below:

\[
\begin{align*}
\Phi_a &= \frac{1}{\sqrt{2}}(\Phi_x - \Phi_z), \\
\Phi_b &= \frac{1}{\sqrt{6}}(\Phi_x - 2\Phi_y + \Phi_z), \\
\Phi_c &= \frac{1}{\sqrt{12}}(\Phi_x + \Phi_y + \Phi_z - 3\Phi_w), \text{ and} \\
\Phi_d &= \frac{1}{2}(\Phi_x + \Phi_y + \Phi_z + \Phi_w).
\end{align*}
\]

Of these normal modes, only \( \{ \Phi_a, \Phi_b, \Phi_c \} \) participate in a nonlinear mode-mixing while the fourth normal mode \( \Phi_d \) remains decoupled from the rest. The Hamiltonian of this device can be written as the sum of the Josephson energies of the three junctions:

\[
H = -E_J \left( \cos \frac{\Phi_x - \Phi_w}{\varphi_0} + \cos \frac{\Phi_y - \Phi_w}{\varphi_0} + \cos \frac{\Phi_z - \Phi_w}{\varphi_0} \right).
\]

Rewriting the Hamiltonian in terms of the normal modes of the device, we get:

\[
H = -E_J \left( 2 \cos \frac{\Phi_x}{\sqrt{2}\varphi_0} \cos \frac{\sqrt{2}\Phi_b + 4\Phi_c}{2\sqrt{3}\varphi_0} + \cos \frac{\sqrt{6}\Phi_b - 2\sqrt{3}\Phi_c}{3\varphi_0} \right).
\]
For small oscillations of the mode amplitudes $\Phi_a, \Phi_b, \Phi_c \ll \varphi_0$, we can expand till the fourth order to arrive at:

\[
H = -3E_J + \frac{E_J}{2\varphi_0^2}(\Phi_a^2 + \Phi_b^2 + 4\Phi_c^2) - \frac{E_J}{144\varphi_0^4} (3\Phi_a^4 + 3\Phi_b^4 + 6\Phi_a^2\Phi_b^2 + 48\Phi_a^2\Phi_c^2 + 48\Phi_b^2\Phi_c^2 - 8\sqrt{2}\Phi_b^2\Phi_c + 24\sqrt{2}\Phi_a\Phi_b\Phi_c)
\]  

(D.8)

Apart from irrelevant frequency renormalization terms quadratic in the mode-amplitudes, at the fourth order, the Hamiltonian has self-Kerr and cross-Kerr terms, together with two interaction terms: $\Phi_b^3\Phi_c$ and $\Phi_a^2\Phi_b\Phi_c$. By applying a pump at a suitable frequency, either of these terms could be made dominant under rotating wave approximation. This is described below.

1. Consider the case when a high-Q resonator mode, with fundamental frequency $\tilde{\omega}_b$ is coupled to this device and a stiff, off-resonant pump at frequency $\tilde{\omega}_c$ is applied. We have introduced the frequencies $\tilde{\omega}_b, \tilde{\omega}_c$ to account for the frequency-shifts that occur due to the aforementioned frequency renormalization, together with the AC Stark shifts due to the applied pump. Further, we require the following frequency constraint:

\[
\tilde{\omega}_c = 3\tilde{\omega}_b.
\]  

(D.9)

To leading order, under rotating wave approximation, the Hamiltonian reduces to:

\[
H/\hbar = \tilde{\omega}_b b^\dagger b + g_{\text{eff}} e^{3i\tilde{\omega}_b t} b^3 + g_{\text{eff}}^* e^{-3i\tilde{\omega}_b t} b^{\dagger 3} - \frac{\chi_{bb}}{2} b^{\dagger 2} b^2,
\]  

(D.10)

where we have absorbed the stiff-pump amplitude in the effective coupling strength $g_{\text{eff}}$ and $\chi_{bb}$ denote the self-Kerr nonlinearities for the mode $b$. In writing the Hamiltonian, we have dropped the part of the Hamiltonian governing the dynamics of the mode $\Phi_a$ since it is never excited. Also, under the stiff pump approximation, there is no dynamics of the pump tone. By applying a strong pump, the three-wave interaction term in the Hamiltonian can be made to dominate over the self-Kerr term.

2. Consider the case when a high-Q resonator mode, with frequency $\tilde{\omega}_a$, and a low-Q resonator mode, with frequency $\tilde{\omega}_b$, are coupled to this device, while a stiff, off-resonant pump at frequency $\tilde{\omega}_c$ is applied. We require

\[
\tilde{\omega}_c = 2\tilde{\omega}_a - \tilde{\omega}_b.
\]  

(D.11)
Under rotating wave and stiff pump approximation, we get the following Hamiltonian:

$$\frac{H}{\hbar} = \tilde{\omega}_a a a^\dagger + \tilde{\omega}_b b b^\dagger + g_{2\text{ph}} e^{2it(2\tilde{\omega}_a - \tilde{\omega}_b)} a^2 b^\dagger + g_{2\text{ph}}^* e^{-2it(2\tilde{\omega}_a - \tilde{\omega}_b)} a^\dagger b^2$$

$$-\frac{\chi_{aa}}{2} a^2 a^2 - \frac{\chi_{bb}}{2} b^2 b^2 - \chi_{ab}(a^\dagger a)(b^\dagger b).$$

(D.12)

This Hamiltonian, together with a resonant drive on the $b$ mode, is crucial to generation of Schrödinger cat states of a given photon-number-parity [42, 77, 119, 125]. While the nonlinear coupling $g_{2\text{ph}}$, together with the resonant drive on the mode $b$, generates the cat states, the inevitable presence of self and cross Kerr nonlinearities distorts the generated state and lowers the fidelity to the target cat state. Note that in contrast to the implementation suggested in [42, 119], this device allows a much larger nonlinear coupling compared to the self and cross Kerr nonlinearities. This allows for a much larger two-photon drive and dissipation rate in comparison to the single photon loss rate for the mode $a$ in this implementation. As a consequence, the Schrödinger cat states generated in this implementation are more resilient to decoherence due to single photon loss (see Chapter A).

### D.3 Josephson Cube Modulator

This section describes a new Josephson circuit device, the Josephson Cube Modulator (JCM). It is comprised of a Josephson junction along each edge of a cube, as shown in Fig. D.2. Following the analysis of the previous section, we define the node-fluxes at each node as follows:

$$\Phi_i = \int_{-\infty}^{t} V_i(t) \, dt, \quad i = x, y, z, w, p, q, r, s.$$  

(D.13)
Figure D.2: Schematic of the Josephson Cube Modulator. Two possible circuits for the Josephson Cube Modulator (JCM) are shown. (a) Twelve Josephson junctions are arranged as shown, one on each edge of the cube. Each vertex of the cube is capacitively coupled to the outside as shown. Seven of the eight normal modes of the device participate in a nonlinear mode-mixing as described in this section. The normal modes for the device are given in Eqs. (D.14)-(D.21) and can be visualized by assigning + or − at each node. (b) Equivalent planar circuit for the JCM. The internal nodes can be accessed using cross-over links from the outside as was done in [92].

The eight normal modes of the device are given as:

\[
\begin{align*}
\Phi_a &= -\Phi_x + \Phi_y + \Phi_z - \Phi_w + \Phi_p - \Phi_q - \Phi_r + \Phi_s, \\
\Phi_b &= \Phi_x - \Phi_y + \Phi_z - \Phi_w - \Phi_p + \Phi_q - \Phi_r + \Phi_s, \\
\Phi_c &= \Phi_x + \Phi_y - \Phi_z - \Phi_w + \Phi_p + \Phi_q - \Phi_r - \Phi_s, \\
\Phi_d &= \Phi_x + \Phi_y + \Phi_z + \Phi_w - \Phi_p - \Phi_q - \Phi_r - \Phi_s, \\
\Phi_e &= \Phi_x - \Phi_y - \Phi_z + \Phi_w + \Phi_p - \Phi_q - \Phi_r + \Phi_s, \\
\Phi_f &= -\Phi_x - \Phi_y + \Phi_z - \Phi_w + \Phi_p + \Phi_q - \Phi_r - \Phi_s, \\
\Phi_g &= \Phi_x - \Phi_y + \Phi_z - \Phi_w + \Phi_p - \Phi_q + \Phi_r - \Phi_s, \\
\Phi_h &= \Phi_x + \Phi_y + \Phi_z + \Phi_w + \Phi_p + \Phi_q + \Phi_r + \Phi_s,
\end{align*}
\]

where we have omitted the unimportant normalization factor (of \(1/\sqrt{8}\)) for each normal mode since it is equal for all of them. Of the normal modes, \(\Phi_i, i = a, \ldots, g\) participate in the nonlinear mode-mixing, while \(\Phi_h\) remains decoupled from the rest.
The Hamiltonian for the device can be written as:

\[
H = -E_J \left( \cos \frac{\Phi_x - \Phi_y}{\phi_0} + \cos \frac{\Phi_y - \Phi_z}{\phi_0} + \cos \frac{\Phi_z - \Phi_w}{\phi_0} + \cos \frac{\Phi_w - \Phi_x}{\phi_0} 
+ \cos \frac{\Phi_p - \Phi_q}{\phi_0} + \cos \frac{\Phi_q - \Phi_r}{\phi_0} + \cos \frac{\Phi_r - \Phi_s}{\phi_0} + \cos \frac{\Phi_s - \Phi_p}{\phi_0} 
+ \cos \frac{\Phi_x - \Phi_p}{\phi_0} + \cos \frac{\Phi_y - \Phi_q}{\phi_0} + \cos \frac{\Phi_z - \Phi_r}{\phi_0} + \cos \frac{\Phi_w - \Phi_s}{\phi_0} \right).
\]

When expressed in terms of the normal modes, the Hamiltonian becomes:

\[
H = -4E_J \cos \frac{\Phi_a}{4\phi_0} \cos \frac{\Phi_b}{4\phi_0} \cos \frac{\Phi_c}{4\phi_0} \cos \frac{\Phi_d}{4\phi_0} \cos \frac{\Phi_g}{4\phi_0} + 4E_J \sin \frac{\Phi_a}{4\phi_0} \sin \frac{\Phi_b}{4\phi_0} \sin \frac{\Phi_c}{4\phi_0} \sin \frac{\Phi_d}{4\phi_0} \sin \frac{\Phi_g}{4\phi_0} 
- 4E_J \cos \frac{\Phi_a}{4\phi_0} \cos \frac{\Phi_c}{4\phi_0} \cos \frac{\Phi_f}{4\phi_0} \cos \frac{\Phi_g}{4\phi_0} + 4E_J \sin \frac{\Phi_a}{4\phi_0} \sin \frac{\Phi_c}{4\phi_0} \sin \frac{\Phi_f}{4\phi_0} \sin \frac{\Phi_g}{4\phi_0} 
- 4E_J \cos \frac{\Phi_b}{4\phi_0} \cos \frac{\Phi_c}{4\phi_0} \cos \frac{\Phi_f}{4\phi_0} \cos \frac{\Phi_g}{4\phi_0} + 4E_J \sin \frac{\Phi_b}{4\phi_0} \sin \frac{\Phi_c}{4\phi_0} \sin \frac{\Phi_f}{4\phi_0} \sin \frac{\Phi_g}{4\phi_0} \right). \tag{D.23}
\]

Thus, the JCM gives rise to three four-wave mixing Hamiltonians, given in each line of Eq. (D.23). In presence of pumps applied at the mode \( \Phi_g \) with appropriately chosen frequencies, this gives rise to three tunable three-wave mixing Hamiltonians.

### D.4 Summary

To summarize, we have presented in this chapter, two Josephson circuit devices which perform nonlinear mode-mixing of microwave resonator modes, in presence of stiff, off-resonant pumps. We have described the normal modes of these devices and outlined the derivation of the Hamiltonians that govern the interaction they participate in. These nonlinear Hamiltonians are crucial to universal quantum computation using continuous variables.
Appendix E

Appendix for Chapter 3

E.1 Schematic of experimental setup

Here, we show a more detailed schematic of the proposed experimental setup (Fig. E.1). Two transmon qubits, Alice and Bob, are initialized in their respective \((|g\rangle + |e\rangle)/\sqrt{2}\) states. They are dispersively coupled to auxiliary cavity modes \(A, B\) respectively. The output from the cavity modes \(A, B\), after propagation on transmission lines, excite the high-Q modes \(a, b\) respectively. The modes \(a, b\) and \(c\) participate in a three-wave mixing interaction, in presence of a stiff, off-resonant pump through the Josephson Four Wave Mixer (see below). The three cavities \(a, b, c\), together with the Josephson Four Wave Mixer, comprise the Josephson Parametric Multiplier (JPM). The outputs of each of the cavities \(a, b, c\) are monitored with homodyne detection, denoted respectively by \(HD_a, HD_b, HD_c\).

E.2 Temporal profile calculation

We show our computation for the dispersive phase shift gathered from dispersive interaction of the propagating ancilla signal mode \(a\) with Alice’s qubit. For brevity, we will compute for the case when the ancilla signal \((a)\) is reflected off Alice’s qubit and proceeds on to be incident on the JPM. One can obtain similar results for the transmission case. The complete cavity-qubit Hamiltonian for the cavity mode \(A\) dispersively coupled to Alice’s qubit can be written as:

\[
H_{A\text{-Alice}} = \hbar \omega_q A^\dagger A + \hbar \omega_q \sigma_z + \chi A^\dagger A \sigma_z, \tag{E.1}
\]
Figure E.1: Schematic of experimental setup for remote entanglement protocol. Two transmon qubits, Alice and Bob, are off-resonantly coupled to auxiliary resonator modes (frequencies) $A(\omega_a)$ and $B(\omega_b)$. The outputs of the modes $A$ and $B$, after propagating through transmission lines, act as inputs to the $a$ and $b$ modes respectively. The modes $a(\omega_a), b(\omega_b)$ and $c(\omega_c)$ participate in a non-linear, three-wave interaction $H_{int}/\hbar = g e^{-i\omega_p t} a b c^\dagger + h.c.$, conditioned on the presence of an off-resonant, stiff pump (in orange) at frequency $\omega_p = \omega_c - \omega_a - \omega_b$. This nonlinear mode mixing arises out of the Josephson Four Wave Mixer (JFWM). The JFWM, together with the resonators of the three interacting modes, comprise the JPM of Fig. (1) in the main text. The outputs of each of the modes $a, b$ and $c$ are monitored with homodyne detection, denoted by $HD_a, HD_b$ and $HD_c$ respectively. (Inset) The JFWM has four nominally identical Josephson junctions and has four mutually orthogonal interacting normal (electrical) modes, respectively corresponding to the modes $a, b, c$ and the stiff, off-resonant pump (see below). The Josephson nonlinearity, together with the off-resonant pump, gives rise to the desired three-wave interaction given by $H_{int}$ under rotating wave approximation.

where $\omega_{a(q)}$ is the frequency of the cavity (qubit) and $\chi$ is the strength of the dispersive interaction between the two. Going to the rotating frame of the cavity at its resonant frequency and that of the qubit, the effective Hamiltonian becomes:

$$H_{disp} = \chi A^\dagger A \sigma_z.$$  \hspace{1cm} (E.2)

We will consider the case when the cavity is excited resonantly. Then, the Langevin equations of motion for the qubit-cavity system (neglecting the dissipation channels of the qubit) can be written
as:
\[
\frac{dA}{dt} = -\frac{\kappa_A}{2}A - i\chi A\sigma_z + \sqrt{\kappa_A}A^\text{in}, \quad A^\text{in} + A^\text{out} = \sqrt{\kappa_A}A, \tag{E.3}
\]

or equivalently,
\[
\frac{dA^\text{out}}{dt} + \frac{\kappa_A}{2}A^\text{out} + i\chi A^\text{out} = -\frac{dA^\text{in}}{dt} + \frac{\kappa_A}{2}A^\text{in} - i\chi A^\text{in}, \tag{E.4}
\]

where \(\kappa_A\) is the cavity decay rate for the mode \(A\).

Denoting the output field amplitude when the qubit is in the excited (ground) state by \(A^\text{out}_e\) (\(g\)), we can write the equations governing the dynamics of each of them:
\[
\frac{dA^\text{out}_e}{dt} + \frac{\kappa_A}{2}A^\text{out}_e + i\chi A^\text{out}_e = -\frac{dA^\text{in}}{dt} + \frac{\kappa_A}{2}A^\text{in} - i\chi A^\text{in}, \tag{E.5}
\]
\[
\frac{dA^\text{out}_g}{dt} + \frac{\kappa_A}{2}A^\text{out}_g - i\chi A^\text{out}_g = -\frac{dA^\text{in}}{dt} + \frac{\kappa_A}{2}A^\text{in} + i\chi A^\text{in}. \tag{E.6}
\]

Without loss of generality, we consider the case when incident field amplitude \(A^\text{in} = A^\text{in}*\), in which case \(A^\text{out}_g = A^\text{out}_e^*\) and thus, Eqn. (E.5) and Eqn. (E.6) are identical to each other. Similar computations can be performed for \(A^\text{in} \in \mathbb{R}\).

Let us define: \(A^\text{out}_g = x + iy\), implying \(A^\text{out}_e = x - iy\). Adding and subtracting Eqns. (E.5) and (E.6), we arrive at:
\[
\begin{align*}
\frac{dx}{dt} + \frac{\kappa_A}{2}x + \chi y &= -\frac{dA^\text{in}}{dt} + \frac{\kappa_A}{2}A^\text{in} \quad \tag{E.7} \\
\frac{dy}{dt} + \frac{\kappa_A}{2}y - \chi x &= \chi A^\text{in}. \quad \tag{E.8}
\end{align*}
\]

We will look for solutions of \(x(t), y(t)\) for \(A^\text{in}(t) = \sqrt{\kappa_a}e^{\kappa_a t/2}\theta(-t)\), where \(\kappa_a\) is the cavity decay rate of the high-Q signal mode of the JPM.

Solving Eqns. (E.7) and (E.8), we arrive at:
\[
\begin{align*}
x(t) &= \frac{\sqrt{\kappa_a}}{(\kappa_a + \kappa_A)^2 + 4\chi^2} \left[ (\kappa_A^2 - \kappa_a^2 - 4\chi^2)e^{\kappa_a t/2}\theta(-t) \\
&\quad + \theta(t)e^{-\kappa_A t/2} \left\{ 2\kappa_A (\kappa_A + \kappa_a) \cos(\chi t) - 4\chi \kappa_A \sin(\chi t) \right\} \right] \tag{E.9} \\
y(t) &= \frac{2\sqrt{\kappa_a\kappa_A}}{(\kappa_a + \kappa_A)^2 + 4\chi^2} \left[ 2\chi e^{\kappa_a t/2}\theta(-t) \\
&\quad + \theta(t)e^{-\kappa_A t/2} \left\{ 2\chi \cos(\chi t) + (\kappa_A + \kappa_a) \sin(\chi t) \right\} \right]. \tag{E.10}
\end{align*}
\]
We require $\kappa_a \ll \kappa_A = 2\chi$, whence:

$$x(t) = \sqrt{\kappa_a} \theta(t) e^{-\kappa_A t/2} \left\{ \cos(\kappa_A t/2) - \sin(\kappa_A t/2) \right\}$$

(E.11)

$$y(t) = \sqrt{\kappa_a} e^{\kappa_A t/2} \theta(-t) + \sqrt{\kappa_a} \theta(t) e^{-\kappa_A t/2} \left\{ \cos(\kappa_A t/2) + \sin(\kappa_A t/2) \right\}.$$  

(E.12)

From the above solution, it is clear that:

1. For $t < 0$, $A_{\text{out}}^g = i\sqrt{\kappa_a} e^{\kappa_A t/2} \theta(-t) = -A_{\text{out}}^e$, which indicates the $\pi$ phase-shift between the output signals when the qubit is in the ground and excited state.

2. For $t > 0$, $A_{\text{out}}^g = \sqrt{\kappa_a} \theta(t) e^{-\kappa_A t/2} e^{i\kappa_A t/2} (1 + i) = A_{\text{out}}^e \ast.$

To find the information content of these pulses, we integrate $|A_{\text{out}}^g|^2$ and $|A_{\text{out}}^e|^2$.

Note that:

$$\int_{-\infty}^{0} |A_{\text{out}}^g|^2 dt = \int_{-\infty}^{0} |A_{\text{out}}^e|^2 dt = 1,$$

(E.13)

$$\int_{0}^{\infty} |A_{\text{out}}^g|^2 dt = \int_{0}^{\infty} |A_{\text{out}}^e|^2 dt = 2\frac{\kappa_a}{\kappa_A} \simeq 0 \text{ for } \kappa_a \ll \kappa_A.$$  

(E.14)

So, for the choice of separation of time-scale, effectively, no photons come out for $t > 0$ and we can treat the output pulse to be a rising exponential wave packet with temporal profile: $\pm i\sqrt{\kappa_a} e^{\kappa_A t/2} \theta(-t)$, where $\pm$ depends on the state of the qubit being in $|g\rangle$ or $|e\rangle$. Numerical simulations have confirmed this result.

Based on the above computation, we can infer the loss of coherence due to the photons coming out for $t > 0$. This loss can be computed to be $1 - e^{-|\alpha_0|^2 \kappa_a / \kappa_A (1 - |K|^2)}$, where $K$ is the overlap between the temporal profiles $A_{\text{out}}^g$ and $A_{\text{out}}^e$ for $t > 0$ and $\alpha_0$ is the coherent state amplitude incident on the cavity. For $\kappa_a \ll \kappa_A$, $|K| \rightarrow 1$ implying no loss of coherence.

To summarize our results, we have shown that a resonant, rising exponential pulse $\sqrt{\kappa_a} e^{\kappa_A t/2} \theta(-t)$ at the input of mode $A$ does indeed give rise to as output a rising exponential pulse $\pm i\sqrt{\kappa_a} e^{\kappa_A t/2} \theta(-t)$ in the $\kappa_a \ll \kappa_A$ regime, where the phase-shift of the reflected signal depends on the state of the qubit.

Note that the phase-shift was calculated in the continuous-wave case and experimentally demonstrated in [135]. Identical set of analysis can be performed for the interaction of the ancilla signal b with Bob’s qubit.
E.3 Quasi-steady State computation under $D(ab)$

The low-Q nature of the mode $c$, together with the interaction Hamiltonian $H_{int}$ (cf. Eq. (4.14)), gives rise to an effective coupled mode dissipation of the form $D(ab)$ with a dissipation rate $\kappa_{2ph} = 4g^2/\kappa_c \gg \kappa_a, \kappa_b$ \cite{77}, where $\kappa_a, \kappa_b$ and $\kappa_c$ are the decay rates of the modes $a, b$ and $c$ respectively.

E.3.1 Even manifold computation

In this subsection, we will describe the computation for the even manifold. Our aim is to compute the steady-state of solution of the Lindblad equation \cite{136–138}:

$$\frac{d\rho_{p=1}^{A_{AB}=1}}{dt} = \kappa_{2ph} D(ab) \rho_{A_{AB}=1}^{p=1},$$

(E.15)

with the initial state: $\rho_{A_{AB}=1}^{p=1}(t = 0) = |\psi_{A_{AB}=1}^{p=1}\rangle \langle \psi_{A_{AB}=1}^{p=1}|$, $|\psi_{A_{AB}=1}^{p=1}\rangle = \frac{1}{\sqrt{2}} (|ee, \alpha, \alpha\rangle + |gg, -\alpha, -\alpha\rangle)$, where for simplicity, we have chosen $\alpha = \beta$. The resulting steady state, denoted by $\rho_{A_{AB}=1}^{p=1}$ can be written as:

$$\rho_{A_{AB}=1}^{p=1} = \frac{1}{2} (\rho_1^{p=1} |ee\rangle \langle ee| + \rho_2^{p=1} |gg\rangle \langle gg| + \rho_3^{p=1} |ee\rangle \langle gg| + \rho_4^{p=1} |gg\rangle \langle ee|),$$

(E.16)
\[
\rho_1^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2) |n,0\rangle \langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2) |0,n\rangle \langle 0,n| \\
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \left\{ |n,0\rangle \langle n,0| + |0,n\rangle \langle 0,n+\mu| + \text{h.c.} \right\} \\
+ \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{(n+\mu)/2}(2|\alpha|^2) \frac{\Gamma(n/2+\mu/2+1)}{\sqrt{n!(\mu)!}} \left\{ |n,0\rangle \langle 0,\mu| + |0,\mu\rangle \langle n,0| \right\}, \quad (E.17)
\]

\[
\rho_2^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2) |n,0\rangle \langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2) |0,n\rangle \langle 0,n| \\
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n \Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \left\{ |n,0\rangle \langle n,0| + |0,n\rangle \langle 0,n+\mu| + \text{h.c.} \right\} \\
+ \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{(n+\mu)/2}(2|\alpha|^2) \frac{(-1)^{n+\mu} \Gamma(n/2+\mu/2+1)}{\sqrt{n!(\mu)!}} \left\{ |n,0\rangle \langle 0,\mu| + |0,\mu\rangle \langle n,0| \right\}, \quad (E.18)
\]

\[
\rho_3^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2) |n,0\rangle \langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2) |0,n\rangle \langle 0,n| \\
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n \Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \left\{ (-1)^n |n,0\rangle \langle n,0| + |n+\mu,0\rangle \langle n,0| \right\} \\
+ \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{(n+\mu)/2}(2|\alpha|^2) \frac{(-1)^{n+\mu} \Gamma(n/2+\mu/2+1)}{\sqrt{n!(\mu)!}} \left\{ (-1)^n |0,\mu\rangle \langle n,0| \right\}, \quad (E.19)
\]

\[
\rho_4^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2) |n,0\rangle \langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2) |0,n\rangle \langle 0,n| \\
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^{n+\mu} \Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \left\{ (-1)^n |n,0\rangle \langle n,0| + |n+\mu,0\rangle \langle n,0| \right\} \\
+ \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} e^{-2|\alpha|^2} I_{(n+\mu)/2}(2|\alpha|^2) \frac{(-1)^{n+\mu} \Gamma(n/2+\mu/2+1)}{\sqrt{n!(\mu)!}} \left\{ (-1)^n |0,\mu\rangle \langle n,0| \right\}, \quad (E.20)
\]

Here \( I_n \) are the modified Bessel functions of the first kind of order \( n \).
E.3.2 Odd manifold computation

In this subsection, we will describe the computation for the odd manifold. Our aim is to compute the steady-state of solution of the Lindblad equation:

$$\frac{d\rho_{ABab}}{dt} = \kappa_{2\text{ph}} D(ab) \rho_{ABab}, \quad (E.21)$$

with the initial state: $\rho_{ABab}^{p=1}(t = 0) = |\psi_{ABab}^{p=1}\rangle\langle\psi_{ABab}^{p=1}|$, $|\psi_{ABab}^{p=1}\rangle = \frac{1}{\sqrt{2}} (|eg,\alpha,-\alpha\rangle + |ge,-\alpha,\alpha\rangle)$, again choosing $\alpha = \beta$ for simplicity. The resulting steady state, denoted by $\rho_{ABab}^{p=1}$ can be written as:

$$\rho_{ABab}^{p=1} = \frac{1}{2} (\rho_1^{p=1} |eg\rangle\langle eg| + \rho_2^{p=1} |ge\rangle\langle ge| + \rho_3^{p=1} |eg\rangle\langle ge| + \rho_4^{p=1} |ge\rangle\langle eg|), \quad (E.22)$$
\[\rho_1^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2)|n,0\rangle\langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2)|0,n\rangle\langle 0,n| + \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|n,0\rangle\langle n+\mu,0| + (-1)^\mu|0,n\rangle\langle 0,n+\mu| + \text{h.c.}\} + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n\Gamma(n/2+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|n,0\rangle\langle 0,\mu| + |0,\mu\rangle\langle n,0|\}, \quad (E.23)\]

\[\rho_2^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2)|n,0\rangle\langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_n(2|\alpha|^2)|0,n\rangle\langle 0,n| + \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|n,0\rangle\langle n+\mu,0| + (-1)^\mu|0,n\rangle\langle 0,n+\mu| + \text{h.c.}\} + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n\Gamma(n/2+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|n,0\rangle\langle 0,\mu| + |0,\mu\rangle\langle n,0|\}, \quad (E.24)\]

\[\rho_3^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2)|n,0\rangle\langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2)|0,n\rangle\langle 0,n| + \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n\Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{(-1)^\mu|n,0\rangle\langle n+\mu,0| + |n+\mu,0\rangle\langle n,0| + |0,n\rangle\langle 0,n+\mu| + (-1)^\mu|0,n\rangle\langle n+\mu,0|\} + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n/2+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|n,0\rangle\langle 0,\mu| + (-1)^n-\mu|0,\mu\rangle\langle n,0|\}, \quad (E.25)\]

\[\rho_4^{p=1} = \sum_{n=0}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2)|n,0\rangle\langle n,0| + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} (-1)^n I_n(2|\alpha|^2)|0,n\rangle\langle 0,n| + \sum_{n=0}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n\Gamma(n+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{(-1)^\mu|n,0\rangle\langle n+\mu,0| + |n+\mu,0\rangle\langle n,0| + |0,n\rangle\langle 0,n+\mu| + (-1)^\mu|0,n\rangle\langle n+\mu,0|\} + \sum_{n=1}^{\infty} e^{-2|\alpha|^2} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n/2+\mu/2+1)}{\sqrt{n!(n+\mu)!}} \{|(-1)^{n-\mu}|n,0\rangle\langle 0,\mu| + |0,\mu\rangle\langle n,0|\}. \quad (E.26)\]

**E.4 Even manifold computation of qubit state after homodyne detection at HD\_a and HD\_b**

**E.4.1 X measurement**

For the detection of X quadratures of the modes a and b, the homodyne spectrum can be modeled by the projective measurement operators of the form \(M_X = |x_a,x_b\rangle\langle x_a,x_b|\), where \(x_a,x_b\) are the outcomes of the integrated homodyne current at HD\_a and HD\_b. The post-measurement state of the
where \( H \) and the final qubit state can be computed from Eq. (E.27) by tracing out the modes \( a \) and \( b \). We use the following definition of the wavefunction of a Fock state in the position basis \([139]\):

\[
\langle x|n \rangle = \left( \frac{2}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(x\sqrt{2}),
\]

(E.28)

where \( H_n \) are Hermite polynomials of order \( n \). Using results obtained in the previous section and omitting a few lines of algebra, the post-measurement qubit state can be written as:

\[
\rho_{AB}(x_a, x_b) = \frac{1}{\lambda_1(x_a, x_b) + \lambda_2(x_a, x_b)} \left( \lambda_1(x_a, x_b) \langle ee| + \lambda_2(x_a, x_b) \langle gg | + \lambda_3(x_a, x_b) \langle ee| + \lambda_4(x_a, x_b) \langle gg | \right),
\]

(E.29)

where

\[
\lambda_1(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2+x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a\sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b\sqrt{2})^2 \right] + \ldots
\]

(E.30)

\[
\lambda_2(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2+x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a\sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b\sqrt{2})^2 \right] + \ldots
\]

(E.31)
\[ \lambda_3(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2 + x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} (-1)^n \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a \sqrt{2})^2 + \sum_{n=1}^{\infty} (-1)^n \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b \sqrt{2})^2 \right] + \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \frac{I_{n+\mu/2}(2|\alpha|^2)}{n!(n+\mu)!2^{n+\mu/2}} \left[ \Gamma(n+\mu/2+1) \right] \right] \]

\[ \lambda_4(x_a, x_b) = \lambda_3(x_a, x_b)^* \quad (E.32) \]

The probability distribution of the outcomes is denoted by:

\[ P(x_a, x_b) = \frac{1}{2} \left[ \lambda_1(x_a, x_b) + \lambda_2(x_a, x_b) \right] \quad (E.33) \]

and the overlap with the Bell-state \(|\phi^+\rangle\) is given by:

\[ \langle \phi^+ | \rho_{AB}(x_a, x_b) | \phi^+ \rangle = \frac{1}{2(\lambda_1(x_a, x_b) + \lambda_2(x_a, x_b))} \left[ \lambda_1(x_a, x_b) + \lambda_2(x_a, x_b) + \lambda_3(x_a, x_b) + \lambda_4(x_a, x_b) \right]. \quad (E.34) \]

These are evaluated for \(\alpha = 0.75\) in the main text.

### E.4.2 Y measurement

In this case, \(M_X \rightarrow M_Y = |y_a, y_b\rangle \langle y_a, y_b|\). We use the following definition of the wavefunction of a Fock state in the momentum basis:

\[ \langle y|n \rangle = \left( \frac{2}{\pi} \right)^{1/4} \frac{(-i)^n}{\sqrt{2^n n!}} e^{-y^2} H_n(y \sqrt{2}), \quad (E.35) \]

where \(H_n\) are Hermite polynomials of order \(n\). Using results obtained in the previous section and omitting a few lines of algebra, the post-measurement qubit state can be written as:

\[ \rho_{AB}(y_a, y_b) = \frac{1}{\lambda_1(y_a, y_b) + \lambda_2(y_a, y_b) + \lambda_3(y_a, y_b) + \lambda_4(y_a, y_b)} \left( \lambda_1(y_a, y_b)|ee\rangle \langle ee| + \lambda_2(y_a, y_b)|gg\rangle \langle gg| + \lambda_3(y_a, y_b)|ee\rangle \langle gg| + \lambda_4(y_a, y_b)|gg\rangle \langle ee| \right). \quad (E.36) \]
A similar set of computation can be done for the odd manifold.

E.5 Odd manifold computation of qubit state after homodyne detection at HD_\alpha and HD_\beta

A similar set of computation can be done for the odd manifold.
E.5.1 X measurements

In this case case, the post-measurement qubit state can similarly be defined as:

$$\rho_{AB}(x_a, x_b) = \frac{1}{\lambda_1(x_a, x_b) + \lambda_2(x_a, x_b) + \lambda_3(x_a, x_b) + \lambda_4(x_a, x_b)} \left( \lambda_1(x_a, x_b)|eg\rangle\langle eg| + \lambda_2(x_a, x_b)|ge\rangle\langle ge| 
+ \lambda_3(x_a, x_b)|eg\rangle\langle ge| + \lambda_4(x_a, x_b)|ge\rangle\langle eg| \right), \quad (E.41)$$

where

$$\lambda_1(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2 + x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a \sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b \sqrt{2})^2 
+ 2 \sum_{n=0}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} \left( H_n(x_a \sqrt{2}) H_{n+\mu}(x_a \sqrt{2}) 
+ (-1)^\mu H_n(x_b \sqrt{2}) H_{n+\mu}(x_b \sqrt{2}) \right) \right], \quad (E.42)$$

$$\lambda_2(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2 + x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a \sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b \sqrt{2})^2 
+ 2 \sum_{n=0}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} \left( (-1)^\mu H_n(x_a \sqrt{2}) H_{n+\mu}(x_a \sqrt{2}) 
+ H_n(x_b \sqrt{2}) H_{n+\mu}(x_b \sqrt{2}) \right) \right], \quad (E.43)$$

$$\lambda_3(x_a, x_b) = \frac{2}{\pi} e^{-2(x_a^2 + x_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} (-1)^n \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_a \sqrt{2})^2 + \sum_{n=1}^{\infty} (-1)^n \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(x_b \sqrt{2})^2 
+ \sum_{n=0}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} (-1)^n \left( (-1)^\mu + 1 \right) \left( H_n(x_a \sqrt{2}) H_{n+\mu}(x_a \sqrt{2}) 
+ H_n(x_b \sqrt{2}) H_{n+\mu}(x_b \sqrt{2}) \right) \right] \right], \quad (E.44)$$

$$\lambda_4(x_a, x_b) = \lambda_3(x_a, x_b)^*.$$  \quad (E.45)
E.5.2 Y measurements

In this case case, the post-measurement qubit state can similarly be defined as:

\[
\rho_{AB}(y_a, y_b) = \frac{1}{\lambda_1(y_a, y_b) + \lambda_2(y_a, y_b)} \left( \lambda_1(y_a, y_b) |e\rangle \langle e| + \lambda_2(y_a, y_b) |g\rangle \langle g| + \lambda_3(y_a, y_b) |e\rangle \langle g| + \lambda_4(y_a, y_b) |g\rangle \langle e| \right),
\]

(E.46)

where

\[
\lambda_1(y_a, y_b) = \frac{2}{\pi} e^{-2(y_a^2 + y_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(y_a \sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(y_b \sqrt{2})^2 \right.
\]

\[
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} \left( i^\mu + (-i)^\mu \right) \left\{ (-1)^n H_n(y_a \sqrt{2}) H_{n+\mu}(y_a \sqrt{2}) - i \frac{(-1)^n}{n!2^{(n+\mu)/2}} \right\},
\]

(E.47)

\[
\lambda_2(y_a, y_b) = \frac{2}{\pi} e^{-2(y_a^2 + y_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(y_a \sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{I_n(2|\alpha|^2)}{2^n n!} H_n(y_b \sqrt{2})^2 \right.
\]

\[
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{(-1)^n \Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} \left( i^\mu + (-i)^\mu \right) \left\{ (-1)^n H_n(y_a \sqrt{2}) H_{n+\mu}(y_a \sqrt{2}) + H_n(y_b \sqrt{2}) H_{n+\mu}(y_b \sqrt{2}) \right\} + \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} I_{n+(n+\mu)/2}(2|\alpha|^2) \frac{(-1)^n \Gamma(n/2 + \mu/2 + 1)}{n!2^{(n+\mu)/2}} \left( i^\mu + (-i)^\mu \right) \left\{ (-1)^n H_n(y_a \sqrt{2}) H_{n+\mu}(y_a \sqrt{2}) + H_n(y_b \sqrt{2}) H_{n+\mu}(y_b \sqrt{2}) \right\},
\]

(E.48)

\[
\lambda_3(y_a, y_b) = \frac{2}{\pi} e^{-2(y_a^2 + y_b^2)} e^{-2|\alpha|^2} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n I_n(2|\alpha|^2)}{2^n n!} H_n(y_a \sqrt{2})^2 + \sum_{n=1}^{\infty} \frac{(-1)^n I_n(2|\alpha|^2)}{2^n n!} H_n(y_b \sqrt{2})^2 \right.
\]

\[
+ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} I_{n+\mu/2}(2|\alpha|^2) \frac{\Gamma(n + \mu/2 + 1)}{n!(n + \mu)!2^{n+\mu/2}} \left( i^\mu + (-i)^\mu \right) \left\{ (1 + (-1)^n) H_n(y_a \sqrt{2}) H_{n+\mu}(y_a \sqrt{2}) + 2i^\mu H_n(y_b \sqrt{2}) H_{n+\mu}(y_b \sqrt{2}) \right\} + \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} I_{n+(n+\mu)/2}(2|\alpha|^2) \frac{(-1)^n \Gamma(n/2 + \mu/2 + 1)}{n!2^{(n+\mu)/2}} \left( i^\mu + (-i)^\mu \right) \left\{ (-1)^n H_n(y_a \sqrt{2}) H_{n+\mu}(y_a \sqrt{2}) + H_n(y_b \sqrt{2}) H_{n+\mu}(y_b \sqrt{2}) \right\},
\]

(E.49)

\[
\lambda_4(y_a, y_b) = \lambda_3(y_a, y_b)^*.
\]

(E.50)
E.6 Comparison of analytical solution with numerical simulations

The analytical results shown above are performed approximating the stochastic evolution in step (TQM) of the protocol with a deterministic evolution. This deterministic evolution leads to the quasi-steady state solution $\rho_{A Bab}^{\pm 1}$ being impure, which results in the post-measurement qubit state $\rho_{AB}$ being impure for a few outcomes. This is reflected in the concurrence shown in Fig. 3, which goes to zero for these outcomes. A full stochastic master equation simulation shows that these are artifacts of the approximation and are absent in the stochastic master equation simulation. Overlap of the qubit states obtained from stochastic master equation solution (denoted by $\rho_{AB}^{\text{SME}}$) with the analytical solutions (denoted by $\rho_{AB}^{\text{ana}}$) is shown in Fig. E.2 for two cases: $\alpha = 0.75$ and $\alpha = 1$. In each of the cases, a sample of 500 trajectories are simulated. For most of the outcomes, the fidelity is well in excess of 90%, while for a certain number of outcomes it is lower and it is checked that these points correspond to the points of zero concurrence. The impurity grows as the $\alpha, \beta$ are increased and this restricts the validity of the analytical model to large $\alpha$. 
Figure E.2: Overlap of the qubit states obtained from analytical and stochastic master equation simulations for $\alpha = 0.75$ and $\alpha = 1$. In each of the cases, a sample of 500 trajectories are simulated. For $\alpha = 0.75$, the overlap between the qubit states obtained from the stochastic master equation simulation ($\rho_{\text{SME}}$) and the corresponding analytical solution $\rho_{\text{ana}}$ is more than 90%, indicating a strong overlap of the model with numerics. For $\alpha = 1$, we do see a few outcomes with lower fidelity, which correspond to the cases when the concurrence goes to zero.
Appendix F

Appendix for Chapter 5

F.1 Protocol to realize a joint-photon-number-modulo-4 measurement

Here, we describe the protocol to perform the joint-photon-number-modulo-4 measurement of two resonator modes. Consider two resonator modes (with annihilation operators $a, b$), which are dispersively coupled to a transmon qubit (whose ground, excited states are denoted by $|g\rangle, |e\rangle$). We require the dispersive coupling strength to be equal $\chi$ for each of the modes $a, b$. The resultant Hamiltonian describing the two cavity modes and the transmon qubit is given by:

$$H_{\text{mod4}} = \omega_q |e\rangle\langle e| + \omega_a a^\dagger a + \omega_b b^\dagger b - \chi (a^\dagger a + b^\dagger b) |e\rangle\langle e|,$$  (F.1)

where $\chi$ is the cross-Kerr coupling of the transmon qubit to the cavity modes. A joint-photon-number-modulo-4 measurement can be performed in the following way. First, a joint-photon-number-modulo-2 measurement is performed [79]. This can be done by exciting the transmon qubit at frequencies $\omega_q - 2k\chi$ where $k \in \mathbb{Z}$, followed by a Z measurement of the transmon. A Z measurement result of $\hat{p}_1 = 1(-1)$ corresponds to a joint-photon-number of the modes $a, b$ being $2k(2k + 1)$.

Second, a measurement is performed that reveals if the joint-photon-number of the arnie and bert modes $\in \{4k, 4k + 1\}$ or not, where $k \in \mathbb{Z}$. This can be done by using the procedure as making the

1. Equal coupling of the transmon qubit to two different cavity modes is challenging to realize experimentally and is not a pre-requisite for making these joint-photon-number measurements. It can be avoided by using higher excited states of the transmon as was demonstrated in the joint-photon-number-modulo-2 measurements of [79].
joint-photon-number-modulo-2 measurement. The only difference is that the transmon qubit is now excited with frequencies \( \omega_q - 4k \chi, \omega_q - (4k + 1) \chi, k \in \mathbb{Z} \). In this case, a Z measurement outcome of \( \tilde{p}_2 = 1 \) corresponds to the joint-photon-number of the modes \( a, b \in \{ 4k, 4k + 1 \} \), while \( \tilde{p}_2 = -1 \) corresponds to the same \( \in \{ 4k + 2, 4k + 3 \} \). From these two measurement outcomes \( \tilde{p}_1, \tilde{p}_2 \), we can infer the joint-photon-number-modulo-4 outcome. For instance, \( \tilde{p}_1 = \tilde{p}_2 = 1 \), the joint-photon-number-modulo-4 outcome \( \lambda = 0 \). Similarly, \( \tilde{p}_1 = -1, \tilde{p}_2 = 1 \Rightarrow \lambda = 1 \), \( \tilde{p}_1 = 1, \tilde{p}_2 = -1 \Rightarrow \lambda = 2 \) and \( \tilde{p}_1 = -1, \tilde{p}_2 = -1 \Rightarrow \lambda = 3 \). Obviously, in an actual experiment, one doesn’t need to send an infinite set of frequencies to make these measurements. The actual number of frequencies depend on the photon-number distributions of the two resonator modes.

F.2 Computation of probability of outcomes and overlap to the Bell-states in absence of imperfections

F.2.1 Implementation using Schrödinger cat states

In this section, we outline the computation of the probability of success and overlap to the different Bell-states. To that end, we start with the state of the four modes: Alice, Bob, arnie and bert:

\[
|\Psi^{p=1}_{ABab}\rangle = \frac{1}{\sqrt{2}} \left( |g, g, C^+, C^+\rangle + |e, e, C^-, C^-\rangle \right),
\]

\[
|\Psi^{p=-1}_{ABab}\rangle = \frac{1}{\sqrt{2}} \left( |g, e, C^+, C^-\rangle + |e, g, C^-, C^+\rangle \right).
\]

The homodyne detection can be modeled as a projection of arnie and bert on \( x \) eigenstates, described by the projection operator: \( \mathcal{M}_X = |x_a, x_b\rangle \langle x_a, x_b| \). Consider the case \( p = 1 \). After the homodyne detection, the un-normalized wave-function for the modes of Alice, Bob, arnie and bert is given by:

\[
\mathcal{M}_X |\Psi^{p=1}_{ABab}\rangle = \frac{1}{\sqrt{2}} \left( \langle x_a|C^+_\alpha \rangle \langle x_b|C^+_\alpha \rangle |g, g\rangle + \langle x_a|C^-_\alpha \rangle \langle x_b|C^-_\alpha \rangle |e, e\rangle \right) |x_a, x_b\rangle.
\]

Using the wave-function in the position basis of an even (odd) Schrödinger cat state:

\[
\langle x|C^\pm_\alpha \rangle = \left( \frac{2}{\pi} \right)^{1/4} N_\pm e^{-x^2-\alpha^2} (e^{2x\alpha} \pm e^{-2x\alpha}),
\]

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we arrive at:

\[
\mathcal{M}_X |\Psi^p=1_{ABab}\rangle = \frac{2}{\sqrt{\pi}} e^{-(x_a^2 + x_b^2)} e^{-2a^2} \left[ \frac{\cosh(2x_a \alpha) \cosh(2x_b \alpha)}{1 + e^{-2a^2}} \right]
\]

\[
|g,g\rangle + \frac{\sinh(2x_a \alpha) \sinh(2x_b \alpha)}{1 - e^{-2a^2}} |e,e\rangle |x_a, x_b\rangle.
\]

The probability distribution of outcomes \(P(x_a, x_b)\) and the resultant state of Alice and Bob \(\rho_{AB}\) are then given by:

\[
P(x_a, x_b) = \frac{1}{2} \text{Tr} \left[ \mathcal{M}_X |\Psi^p=1_{ABab}\rangle \langle \Psi^p=1_{ABab}| \mathcal{M}_X^\dagger \right], \tag{F.5}
\]

\[
\rho_{AB} = \frac{\text{Tr}_{ab} \left[ \mathcal{M}_X |\Psi^p=1_{ABab}\rangle \langle \Psi^p=1_{ABab}| \mathcal{M}_X^\dagger \right]}{\text{Tr} \left[ \mathcal{M}_X |\Psi^p=1_{ABab}\rangle \langle \Psi^p=1_{ABab}| \mathcal{M}_X^\dagger \right]}, \tag{F.6}
\]

giving rise to: Eqs. (5.7), (5.8). Note the factor of 1/2, which arises in the total probability of outcomes due to the fact that the \(p = 1\) outcome happens with a probability 1/2. The calculation for \(p = -1\) can be done in an analogous fashion.

### F.2.2 Implementation using mod 4 cat states

In this section, we outline the calculation for the probability of success and the overlap to the Bell-states when the ancilla qubits are encoded in mod 4 cat states. The state of Alice, Bob, arnie and Bert, following the joint-photon-numbermodulo-4 measurement can be written as [Eq. (5.12), (5.13)]:

\[
|\Psi^\lambda=0_{ABab}\rangle = \frac{1}{\sqrt{2}} (|g, g, C^0_{\alpha \text{mod}4}, C^0_{\alpha \text{mod}4}\rangle + |e, e, C^2_{\alpha \text{mod}4}, C^2_{\alpha \text{mod}4}\rangle)
\]

\[
|\Psi^\lambda=2_{ABab}\rangle = \frac{1}{\sqrt{2}} (|g, g, C^0_{\alpha \text{mod}4}, C^0_{\alpha \text{mod}4}\rangle + |e, g, C^2_{\alpha \text{mod}4}, C^2_{\alpha \text{mod}4}\rangle). \tag{F.7}
\]

To compute resultant states of Alice and Bob after the subsequent homodyne detection of arnie and bert, we use the following definitions of the mod 4 cats:

\[
|C^{\lambda \text{mod}4}_{\alpha}\rangle = N_\lambda \left( |C^{(-\lambda)}_{\alpha}\rangle + (-i)^\lambda |C^{(-\lambda)}_{\alpha}\rangle \right), \tag{F.8}
\]
where \( \tilde{N}_\lambda = \left[ 2 + 2(-i)^\lambda \{ e^{i|\alpha|^2} + (-1)^\lambda e^{-i|\alpha|^2} \}/ \{ e^{i|\alpha|^2} + (-1)^\lambda e^{-i|\alpha|^2} \} \right]^{-\frac{1}{2}}, \lambda \in \{ 0, 1, 2, 3 \} \) and \((-i)^\lambda = +(-)\) for even (odd) \( \lambda \). This leads to:

\[
\langle x | C_\alpha^{\lambda \mod 4} \rangle = \tilde{N}_\lambda (\langle x | C_\alpha^+ \rangle + i^\lambda \langle x | C_{i\alpha}^+ \rangle), \quad \lambda = 0, 2
\]

where \( \langle x | C_\alpha^+ \rangle \) and \( \langle x | C_{i\alpha}^+ \rangle \) are given by:

\[
\langle x | C_\alpha^+ \rangle = 2 \left( \frac{2}{\pi} \right)^{1/4} \tilde{N}_+ e^{-x^2 - \alpha^2} \cosh(2\alpha x), \quad (F.9)
\]

\[
\langle x | C_{i\alpha}^+ \rangle = 2 \left( \frac{2}{\pi} \right)^{1/4} \tilde{N}_+ e^{-x^2} \cos(2\alpha x). \quad (F.10)
\]

Thus, we arrive at:

\[
\mathcal{M}_X | \Psi_{ABab}^{\lambda=0} \rangle = \frac{1}{\sqrt{2}} \left( \langle x_a, x_b | C_{0\mod 4} \rangle \langle g, g | \rangle + \langle x_a, x_b | C_{2\mod 4} \rangle \langle e, e | \rangle \right) | x_a, x_b \rangle
\]

\[
= \frac{4}{\sqrt{\pi}} \tilde{N}_+^2 e^{-(x_a^2 + x_b^2)} \left[ \tilde{N}_0^2 F_0(x_a) F_0(x_b) | g, g \rangle + \tilde{N}_2^2 F_2(x_a) F_2(x_b) | e, e \rangle \right] | x_a, x_b \rangle, \quad (F.11)
\]

where \( F_\lambda(x) = e^{-\alpha^2} \cosh(2\alpha x) + i^\lambda \cos(2\alpha x) \). This leads to the probability distribution and the resultant density matrix for Alice and Bob through the relations:

\[
P(x_a, x_b) = \frac{1}{2} \text{Tr} \left[ \mathcal{M}_X | \Psi_{ABab}^{\lambda=0} \rangle \langle \Psi_{ABab}^{\lambda=0} | \mathcal{M}_X^\dagger \right], \quad (F.12)
\]

\[
\rho_{AB} = \frac{\text{Tr}_{ab} \left[ \mathcal{M}_X | \Psi_{ABab}^{\lambda=0} \rangle \langle \Psi_{ABab}^{\lambda=0} | \mathcal{M}_X^\dagger \right]}{\text{Tr} \left[ \mathcal{M}_X | \Psi_{ABab}^{\lambda=0} \rangle \langle \Psi_{ABab}^{\lambda=0} | \mathcal{M}_X^\dagger \right]}. \quad (F.13)
\]

Similar set of calculations can be done for the outcome \( \lambda = 2 \).

**F.3 Computation of propagating qubit-photon states in presence of imperfections**

**F.3.1 Implementation using Schrödinger cat states**

First, we describe the computation of the state of Alice and arnie after their entangled qubit-photon states decohere as they propagate through the transmission line. The initial state of Alice and arnie
is given by [Eq. (5.18)]:

$$|\Psi_{Aa}\rangle = \frac{1}{\sqrt{2}} \sum_{j,\mu=0}^{1} \mathcal{N}_j (-1)^{j\mu} |j, (-1)^{\mu} \bar{\alpha}\rangle.$$  

(F.14)

To compute the final state after attenuation losses, first we introduce an auxiliary mode $a'$, initialized to vacuum. Then, we model the losses by passing the joint system of Alice, arnie and $a'$ through a beam-splitter with transmission probability $\eta_1$. Thus, the state evolves according to:

$$|\Psi_{Aa}\rangle \otimes |0\rangle \rightarrow |\tilde{\Psi}_{Aa}\rangle$$

$$|\tilde{\Psi}_{Aa}\rangle = \frac{1}{\sqrt{2}} \sum_{j,\mu=0}^{1} \mathcal{N}_j (-1)^{j\mu} |j, (-1)^{\mu} \bar{\alpha}, (-1)^{\mu} \epsilon\rangle,$$  

(F.15)

where $\bar{\alpha} = \sqrt{\eta_1} \alpha$ and $\epsilon = \sqrt{1-\eta_1} \alpha$. Thus, the density matrix for the modes Alice, arnie and $a'$ can be written as:

$$\tilde{\rho}_{Aa} = \frac{1}{2} \sum_{j,j',\mu,\mu'=0}^{1} \mathcal{N}_j \mathcal{N}_{j'} (-1)^{j\mu} \left( |j, (-1)^{\mu} \bar{\alpha}\rangle \langle j' , (-1)^{\mu'} \bar{\alpha}|\right) (-1)^{\epsilon} (-1)^{\epsilon'}$$

$$\Rightarrow \rho_{Aa} = \frac{1}{2} \sum_{j,j',\mu,\mu'=0}^{1} \mathcal{N}_j \mathcal{N}_{j'} (-1)^{j\mu} e^{-\epsilon^2 (1-(-1)^{\mu+\mu'})} \left( |j, (-1)^{\mu} \bar{\alpha}\rangle \langle j' , (-1)^{\mu'} \bar{\alpha}|\right)$$  

(F.16)

where $\mu = \{\mu, \mu'\}, j = \{j, j'\}$ and in the last line, we have traced out the auxiliary mode $a'$. Here, $e^{-\epsilon^2 (1-(-1)^{\mu+\mu'})}$ shows explicitly the loss of coherence due to the loss of information to the environment. At this point, we need to re-express the $|(-1)^{\mu} \bar{\alpha}\rangle$ in the eigenbasis of the measurement operator: joint-photon-number-modulo-2. To that end, we use:

$$|(-1)^{\mu} \bar{\alpha}\rangle = \frac{1}{2} \sum_{k=0}^{1} \frac{(-1)^{\mu k}}{\sqrt{N_k}} C_{\bar{\alpha}}^{(-k)}$$  

(F.17)

where $\sqrt{N_j} = 1/\sqrt{2(1+(-1)^{j}e^{-2|\bar{\alpha}|^2})}$. Using Eqs. (F.16), (F.17), we arrive at the density matrix for Alice and arnie:

$$\rho_{Aa} = \frac{1}{8} \sum_{Aa} \sum_{j,j'=0}^{1} \sum_{k,k'=0}^{1} \sum_{\mu,\mu'=0}^{1} \mathcal{N}_j \mathcal{N}_{j'} (-1)^{j+k} e^{-\epsilon^2 (1-(-1)^{\mu+\mu'})} \left| j, C_{\bar{\alpha}}^{(-k)} \right\rangle \left\langle j', C_{\bar{\alpha}}^{(-k')} \right|$$  

(F.18)

where $\sum_{Aa} = \sum_{j,j'=0}^{1} \sum_{k,k'=0}^{1} \sum_{\mu,\mu'=0}^{1} k = \{k, k'\}$.

Similar calculations can be done for the entangled states of Bob and bert, yielding:

$$\rho_{Bb} = \frac{1}{8} \sum_{Bb} \sum_{l,l'=0}^{1} \sum_{m,m'=0}^{1} \mathcal{N}_l \mathcal{N}_{l'} (-1)^{l+m} e^{-\epsilon^2 (1-(-1)^{\mu+\mu'})} \left| l, C_{\bar{\alpha}}^{(-m)} \right\rangle \left\langle l', C_{\bar{\alpha}}^{(-m')} \right|$$  

(F.19)
\[ \sum_{Bb} = \sum_{l,l'} \sum_{m,m'} \sum_{\nu, \nu'} \nu = \{ \nu, \nu' \}, l = \{ l, l' \} \text{ and } m = \{ m, m' \}. \]

The tensor product of \( \rho_{Aa} \) and \( \rho_{Bb} \) gives Eq. (5.19).

### F.3.2 Implementation using mod 4 cat states

First, we describe the computation of the qubit-photon states of Alice and arnie after propagation through the transmission line. The starting point is Eq. (5.23):

\[
|\Psi_{Aa}\rangle = \frac{1}{\sqrt{2}} \sum_{j,\mu,\nu} \tilde{N}_j \tilde{N}_j (-1)^{\nu j} |j, (-1)^{\mu i} \alpha\rangle.
\] (F.20)

To compute the entangled qubit-photon state when it arrives at the joint-photon-number-modulo-4 measurement apparatus, we use the approach outlined in Appendix. F.3.1, introducing an auxiliary mode \( a' \) in vacuum, computing the resultant state of Alice, arnie and \( a' \) as it passes through a beam-splitter of transmission probability \( \eta \) and subsequently, tracing out the mode \( a' \). Following the notation in Appendix. F.3.1, we find that:

\[
|\bar{\Psi}_{Aa}\rangle = \frac{1}{\sqrt{2}} \sum_{j,\mu,\nu} \tilde{N}_j \tilde{N}_j (-1)^{\nu j} |j, (-1)^{\mu i} \bar{\alpha}, (-1)^{\mu i} \epsilon\rangle,
\]

where \( \bar{\alpha} = \sqrt{\eta} \alpha \) and \( \epsilon = \sqrt{1 - \eta} \alpha \). Thus, the density matrix for the modes Alice, arnie and \( a' \) can be written as:

\[
\tilde{\rho}_{Aa} = \frac{1}{2} \sum_{\{ j \}} \tilde{N}_j \tilde{N}_j (-1)^{\nu j} |j, (-1)^{\mu i} \bar{\alpha}\rangle \langle j, (-1)^{\mu i} \epsilon| (-1)^{\mu i} \epsilon \rangle
\]

\[
= \rho_{Aa} = \frac{1}{2} \sum_{\{ j \}} \tilde{N}_j \tilde{N}_j (-1)^{\nu j} e^{-2(1-(-1)^{\mu i} \nu)} \langle j, (-1)^{\mu i} \bar{\alpha}\rangle \langle j', (-1)^{\mu i} \epsilon| (-1)^{\mu i} \epsilon \rangle,
\]

where \( \sum_{\{ j \}} = \sum_{j,j'=0} \sum_{\mu,\mu'=0} \sum_{\nu,\nu'=0} \nu = \{ \nu, \nu' \}, j = \{ j, j' \} \) and in the last line, we have traced out the auxiliary mode \( a' \). In the next step, we express the above equation in the eigenbasis of the joint-photon-number-modulo-4 measurement. To that end, we use:

\[
|(-1)^{\mu i} \bar{\alpha}\rangle = \frac{1}{4} \sum_{\gamma=0}^{3} \frac{1}{\tilde{N}_\gamma \tilde{N}_\gamma} |C_{\gamma \text{mod}4}^{\nu \mu i} \rangle = \frac{1}{4} \sum_{\gamma=0}^{3} \frac{(-1)^{\mu i} \nu \gamma}{\tilde{N}_\gamma \tilde{N}_\gamma} |C_{\bar{\alpha}}^{\gamma \text{mod}4} \rangle,
\] (F.21)

where \( \tilde{N}_\gamma, \tilde{N}_\gamma \) can obtained from the definitions of \( \tilde{N}_\gamma, \tilde{N}_\gamma \) (cf. Secs. 5.4.1, 5.4.2) by making the substitution \( \alpha \rightarrow \bar{\alpha} \) and the last line follows from the definition of mod 4 cats (see Sec. 2.2 of [42]).
Combining the last two equations results in:

\[
\rho_{\text{AA}} = \frac{1}{25} \sum_{\text{Aa}} \frac{\tilde{N}_{2j} \tilde{N}_{2j'} \tilde{N}_{2j''}}{\tilde{N}_{2j} \tilde{N}_{2j'} \tilde{N}_{2j''}} (-1)^{\nu j + \nu' j'} e^{-\epsilon^2 (1 - (-1)^{\mu j + \mu' j'}) \bar{N}_{2j} \bar{N}_{2j'} \bar{N}_{2j''}} \langle j, C_{\tilde{\alpha}} \rangle |j, C_{\tilde{\alpha}}' \rangle,
\]

where \( \sum_{\text{Aa}} = \sum_{j,j'=0}^{1} \sum_{\mu,\mu'=0}^{1} \sum_{\nu,\nu'=0}^{1} \sum_{\gamma,\gamma'=0}^{3} \). Here, we have used, as before, the following definitions: \( j = \{ j, j' \}, \mu = \{ \mu, \mu' \}, \nu = \{ \nu, \nu' \} \) and \( \gamma = \{ \gamma, \gamma' \} \). Similar calculations can be done for Bob and Bert, yielding:

\[
\rho_{\text{BB}} = \frac{1}{25} \sum_{\text{Bb}} \frac{\tilde{N}_{2k} \tilde{N}_{2k'} \tilde{N}_{2k''}}{\tilde{N}_{2k} \tilde{N}_{2k'} \tilde{N}_{2k''}} (-1)^{\psi k + \psi' k'} e^{-\epsilon^2 (1 - (-1)^{\phi k + \phi' k'}) \bar{N}_{2k} \bar{N}_{2k'} \bar{N}_{2k''}} \langle j, C_{\tilde{\alpha}} \rangle |j, C_{\tilde{\alpha}}' \rangle,
\]

where \( \sum_{\text{Bb}} = \sum_{k,k'=0}^{1} \sum_{\phi,\phi'=0}^{1} \sum_{\psi,\psi'=0}^{1} \sum_{\delta,\delta'=0}^{3} \sum_{\phi,\phi'=0}^{1} \sum_{\psi,\psi'=0}^{1} \sum_{\delta,\delta'=0}^{3} \). The tensor product of \( \rho_{\text{AA}} \) and \( \rho_{\text{BB}} \) gives us Eq. (5.24).

After the homodyne detection of Arnie and Bert, corresponding to the joint-photon-number-modulo-4 measurement outcomes \( \lambda = 1, 3 \), the probability of success and the overlap to the Bell-states \( |\phi^\pm\rangle, |\psi^\pm\rangle \) is shown in Fig. F.1.
Figure F.1: Probability distribution $\bar{P}^{\lambda}(q_a, q_b)$ of outcomes of the homodyne measurements of arnie and bert and resulting overlap of Alice and Bob’s joint density matrix $\rho_{AB}$ with the Bell-states $|\phi^\pm\rangle = (|g,g\rangle \pm |e,e\rangle)/\sqrt{2}$, $|\psi^\pm\rangle = (|g,e\rangle \pm |e,g\rangle)/\sqrt{2}$ is shown. We choose $\alpha = 1$ and $\eta_1 = \eta_2 = 0.8$ and show the cases $\lambda = 1, 3$ (see Appendix. F.3.2 for $\lambda = 1, 3$). The top (bottom) left panel shows the probability of outcomes for the joint-photon-number-modulo-4 outcome to be $\lambda = 1(3)$. Corresponding overlaps to the Bell-states $|\psi^\pm\rangle (|\phi^\pm\rangle)$ are plotted in the top (bottom) center and top (bottom) right panels. The overlaps to the even (odd) Bell-states for $\lambda = 1(3)$ are not shown for brevity.
Bibliography


